Symplectic Geometry and Quantisation

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Contents

1 Symplectic Essentials

Definition 1.1. Let V be a finite-dimensional real vector space. A 2-form $\omega \in \bigwedge^2 V^*$ is non-degenerate if for all non-zero $u \in V$ there exists some $v \in V$ such that $\omega(u, v)$ is non-zero. We then say ω is a symplectic form, and (V, ω) a symplectic vector space.

Proposition 1.2. *On a symplectic vector space (V, ω), there exists a basis such that in this basis (as row vectors) we have $\omega(u, v) = u\Omega v^{\mathrm{T}}$ where Ω is a matrix with block form*

$$\begin{bmatrix} 0 & Id \\ -Id & 0 \end{bmatrix}.$$

Proof. See [2] theorem 1.1. □

As an immediate corollary, we have that the vector space V must be even dimensional.

Example 1.3. The standard example is $\mathbb{R}^{2n}$ with the 2-form

$$\omega_0 = \sum_{j=1}^{n} \mathrm{d}x_j \wedge \mathrm{d}y_j.$$

By proposition 1.2, any $2n$-dimensional symplectic vector space can be given a basis such that the symplectic form is of the form ω_0 (by identifying the tangent space to the vector space with the vector space). Then we see that

$$\omega_0^n = \frac{1}{n!}\,\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n$$

which is a non-zero top-form.

Definition 1.4. Let M be a finite-dimensional smooth manifold. We say a 2-form $\omega \in \Omega^2(M)$ is non-degenerate if for every point $p \in M$, $\omega_p \in \bigwedge^2 \mathrm{T}^* M$ is non-degenerate. Furthermore, we say ω is a symplectic form, and (M, ω) a symplectic manifold, if ω is closed.

It might seem restrictive to only consider closed forms however we shall see in §2 that the closure condition is crucial as otherwise the Hamiltonian vector fields will not preserve the symplectic form. In a sense, it prohibits the local geometry from changing too much across the vector field. For example, on R^2 while $f\,\mathrm{d}x \wedge \mathrm{d}y$ is non-degenerate for a non-vanishing smooth function f, it will only be closed if f is constant. Why not then force the 2-form to be exact as well? That would be far too restrictive, precluding the manifold from being compact.

Proposition 1.5 ([1] II.1.6)**.** *If M is a compact manifold, then there exists no 2-form on M that is both non-degenerate and exact.*

As mentioned before, a vector space must be even-dimensional to be symplectic, thus a symplectic manifold (M, ω) must also be even dimensional. In example 1.3 we showed that ω_0^n is a non-zero top-form, therefore if ω is a symplectic form, ω^n is a non-vanishing top form, thus the manifold must be orientable. We usually scale this top-form by a factor of $\frac{1}{n!}$.

Definition 1.6. Let W be a linear subspace of a symplectic vector space (V, ω). Then we define its symplectic complement as the subspace

$$W^{\perp} = \{u \in V : \omega(u, v) = 0 \text{ for all } v \in W\}.$$

Note that unlike the orthogonal complement $W \cap W^{\perp}$ is not necessarily trivial. We say a subspace W is:

- symplectic if $W \cap W^{\perp} = \{0\}$,
- isotropic if $W \subseteq W^{\perp}$,
- coisotropic if $W^{\perp} \subseteq W$,
- Lagrangian if W is both isotropic and coisotropic, i.e. $W = W^{\perp}$.

Now for a few useful lemmas about the symplectic complement.

Lemma 1.7. *Let (V, ω) be a symplectic vector space. If W is a subspace of V then*

$$\dim W + \dim W^{\perp} = \dim V.$$

Proof. Consider the map

$$\begin{aligned} f \colon V &\mapsto W^* \\ v &\mapsto \omega(v, \cdot)|_W. \end{aligned}$$

As ω is non-degenerate the map

$$\begin{aligned} V &\mapsto V^* \\ v &\mapsto \omega(v, \cdot) \end{aligned}$$

is injective and thus by rank-nullity is surjective. Thus f is surjective and its kernel is clearly $W^{\perp}$ so again by rank-nullity

$$\dim W + \dim W^{\perp} = \dim V.$$

□

Clearly $W \subseteq W^{\perp}$ and the above lemma implies that this is an equality.

Lemma 1.8 ([2] 23.3). *Let (V, ω) be a symplectic vector space and W a coisotropic subspace. Then ω descends to a canonical symplectic form on $W/W^{\perp}$.*

Proof. For $u, v \in W$ define $\Omega(u+W^{\perp}, v+W^{\perp}) = \omega(u, v)$. This is well defined: let $a, b \in W^{\perp}$ then

$$\omega(u+a, v+b) = \omega(u, v) + \omega(u, b) + \omega(a, v) + \omega(a, b) = \omega(u, v)$$

as $W^{\perp} \subseteq W$. Now suppose that $u \in W$ and $\omega(u, v) = 0$ for all $v \in W$. Then $u \in W^{\perp}$ thus Ω is non-degenerate. □

If (M, ω) is a symplectic manifold, and $S \subseteq M$ an immersed or embedded submanifold, we say S is a symplectic submanifold if for every $p \in S$, the subspace $\mathrm{T}_p S \leq \mathrm{T}_p M$ is symplectic. Similarly for isotropic, coisotropic, and Lagrangian.

Example 1.9 ([2, 11]). Let M be an (possibly odd dimensional) arbitrary n-manifold. Then we want to show that its cotangent bundle $\pi : \mathrm{T}^* M \to M$ has a natural structure of a symplectic manifold. Suppose $x_1, \ldots, x_n$ are local coordinates on some chart U for M and $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ the associated cotangent coordinates on $\mathrm{T}^* U$ for $\mathrm{T}^* M$. Then consider the element of $\mathrm{T}^* \mathrm{T}^* U$ given by

$$\alpha = \sum \xi_i \, \mathrm{d}x_i.$$

Then

$$-\mathrm{d}\alpha = \sum \mathrm{d}x_i \wedge \mathrm{d}\xi_i$$

which we know is a symplectic form on the image of $\mathrm{T}^* U$ in $\mathbb{R}^{2n}$. How do we define this on all of $\mathrm{T}^* M$? We could show that the definitions coincide on any two coordinate charts however we can construct a coordinate independent definition. Notice that α, a 1-form on $\mathrm{T}^* U$, has a similar form to a 1-form on U. Then in the coordinates (x, ξ) the map $\pi \colon \mathrm{T}^* U \to U$ is given by $\pi(x, \xi) = x$. Thus $\mathrm{d}\pi^*_{(x,\xi)} \colon \mathrm{T}^*_x U \to \mathrm{T}^*_{(x,\xi)} \mathrm{T}^* U$ maps $\mathrm{d}x_i$ to $\mathrm{d}x_i$ and hence

$$\mathrm{d}\pi^*_{(x,\xi)}(\xi_x) = \mathrm{d}\pi^*_{(x,\xi)}(\sum \xi_i \, \mathrm{d}x_i) = \sum \xi_i \, \mathrm{d}x_i = \alpha_{(x,\xi)}.$$

Thus α is coordinate independent, and is called the tautological 1-form on $\mathrm{T}^* M$. The symplectic form $\omega = -\mathrm{d}\alpha$ is called the canonical symplectic form on $\mathrm{T}^* M$. The given the 'tautological' construction of α and ω it is unsurprising that these forms are natural in the sense that a diffeomorphism between two manifolds M and N lifts to a symplectomorphism between $\mathrm{T}^* M$ and $\mathrm{T}^* N$ (see [2]). Can we give the tangent bundle a symplectic structure? One can always choose a Riemannian metric on M and use this to give a diffeomorphism between $\mathrm{T}^* M$ and $\mathrm{T} M$ and then pullback the symplectic form on $\mathrm{T}^* M$ however this will not be natural — the diffeomorphism depends on the choice of Riemannian metric.

We record some ideas about Lie group actions (mostly from [1, 11]). Suppose G is a Lie group that acts smoothly on a manifold M. Then for a point $p \in M$ denote the orbit space of p as $G \cdot p$, and its stabiliser (a Lie subgroup of G) as G_p.

Proposition 1.10 ([1] chapter I)**.** *For each point $p \in M$ we have a smooth map*

$$\begin{aligned} f_p \colon G &\to M \\ g &\mapsto g \cdot p \end{aligned}$$

mapping G onto the orbit of p. Its derivative at the identity gives us a linear map

$$\mathrm{d}f_p|_e \colon \mathfrak{g} \to \mathrm{T}_p M$$

so that for each vector $X \in \mathfrak{g}$ we have a smooth vector field $\underline{X}_p := \mathrm{d}f_p|_e(X)$ with flow $\exp(tX) \cdot p$. Furthermore, for $X, Y \in \mathfrak{g}$

$$\underline{[X, Y]} = -[\underline{X}, \underline{Y}].$$

Proof. See [1][1] $\square$

Proposition 1.11. *Suppose G is compact[2] and connected. Then each orbit $G \cdot p$ is an embedded submanifold of M which is G-equivariantly diffeomorphic to G/G_p. Infinitesimally, we have that $\mathrm{T}_p(G \cdot p) \cong \mathfrak{g}/\mathfrak{g}_p$ as vector spaces, where $\mathfrak{g}_p$ is the Lie algebra of G_p. We also have that*

$$\mathfrak{g}_p = \{X \in \mathfrak{g} : \underline{X}_p = 0\}.$$

Proof. See [1] Corollary I. $\square$

[1] Except they state that $\underline{[X, Y]} = [\underline{X}, \underline{Y}]$, however this is incorrect for left Lie group actions, see [11] Theorem 20.18. We actually have a Lie algebra *anti*-homomorphism between $\mathfrak{g}$ and $\mathfrak{X}(M)$.

[2] This also holds if the action is proper.

2 Moment Maps

2.1 Hamiltonian Vector Fields

Proposition 2.1. *Let (M, ω) be a symplectic manifold. Then ω induces a vector bundle isomorphism between $\mathrm{T}\, M$ and $\mathrm{T}^*\, M$.*

Proof. Define a map $\hat{\omega} : \mathrm{T}\, M \to \mathrm{T}^*\, M$ by $\hat{\omega}(v) : u \mapsto \omega_x(u, v)$ for $v \in \mathrm{T}_x\, M$. This is clearly a bundle map and smoothness of $\hat{\omega}$ follows from smoothness of ω. Finally, non-degeneracy of ω implies that $\hat{\omega}$ is bijective at each point and thus a vector bundle isomorphism. □

In particular, $\hat{\omega}$ is a linear isomorphism between $\mathfrak{X}(M)$ and $\mathrm{T}^*(M)$. Thus if we have a smooth function $f \in \mathrm{C}^\infty(M)$, its derivative $\mathrm{d}f$ lies in $\mathfrak{X}^*(M)$ and thus ω gives us a unique vector field $X_f \in \mathfrak{X}(M)$ by $X_f := \hat{\omega}^{-1}(\mathrm{d}f)$. Equivalently, we have that $\iota_{X_f}\omega = -\,\mathrm{d}f$.

Definition 2.2. Let f be a smooth function on M. Then the unique vector vector field X_f on M such that $\iota_{X_f}\omega = -\,\mathrm{d}f$ is called the symplectic gradient or the Hamiltonian vector field of f, and f is called the Hamiltonian function for X_f. We denote the vector space[3] of Hamitonian vector fields on M by $\mathcal{H}(M)$.

Recall that any smooth vector field X on M can be viewed as a derivation of smooth functions on M. Suppose then we have two smooth functions f and g on M. Then we can apply the Hamiltonian vector field of f to g to get another smooth function $X_f g$, which we shall call the Poisson bracket $\{f, g\}$. Geometrically, this measures the rate of change of g along the Hamiltonian flow of f. This Poisson bracket interacts well with the Lie bracket as the next proposition shows.

Proposition 2.3. *Let X, Y be the Hamiltonian vector fields of f, g respectively. Then the Hamiltonian vector field of $\{f, g\}$ is $[X, Y]$.*

Proof. First we prove a lemma relating interior products and Lie brackets.

Lemma 2.4. *For vector fields X and Y and a differential form γ we have that*

$$\iota_{[X,Y]}\gamma = [\mathcal{L}_X, \iota_Y]\gamma := \mathcal{L}_X \iota_Y \gamma - \iota_Y \mathcal{L}_X \gamma.$$

[3] the fact that this is a vector space follows immediately from the $\mathbb{R}$-linearity of ι and d

Proof of Lemma 2.4. For a 0-form f we have that $\iota_Y f$, $\iota_X \mathcal{L} f$, and $\iota_{[X,Y]} f$ are all zero as the interior product of a function is zero by definition. So the lemma is true for 0-forms. Now let γ be a 1-form. We have:

$$\begin{aligned}
[\mathcal{L}_X, \iota_Y]\gamma &= \mathcal{L}_X \iota_Y \gamma - \iota_Y \mathcal{L}_X \gamma \\
&= X(\gamma(Y)) - \iota_Y \iota_X \,\mathrm{d}\gamma - \iota_Y \,\mathrm{d}\iota_X \gamma \\
&= X(\gamma(Y)) - (X(\gamma(Y)) - Y(\gamma(X)) - \gamma([X,Y])) - Y(\gamma(X)) \\
&= \gamma([X,Y]) = \iota_{[X,Y]}\gamma \quad .
\end{aligned}$$

Thus the lemma is true for 1-forms. Next we notice that for a k-form α and an l-form β we have

$$\begin{aligned}
\mathcal{L}_X(\alpha \wedge \beta) &= (\mathcal{L}_X \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X \beta) && \text{and} \\
\iota_Y(\alpha \wedge \beta) &= (\iota_Y \alpha) \wedge \beta + (-1)^k \alpha \wedge (\iota_Y \beta) && \text{gives us} \\
[\mathcal{L}_X, \iota_Y](\alpha \wedge \beta) &= ([\mathcal{L}_X, \iota_Y]\alpha) \wedge \beta + \alpha \wedge ([\mathcal{L}_X, \iota_Y]\beta) \quad .
\end{aligned}$$

Thence by induction we have proved the lemma. □

Then

$$\begin{aligned}
&\iota_{[X,Y]}\omega \\
&= [\mathcal{L}_X, \iota_Y]\omega && \text{by Lemma 2.4} \\
&= \mathrm{d}\iota_X \iota_Y \omega + \iota_X \,\mathrm{d}\iota_Y \omega - \iota_Y \,\mathrm{d}\iota_X \omega - \iota_Y \iota_X \,\mathrm{d}\omega && \text{by Cartan's} \\
&= \mathrm{d}\iota_X \iota_Y \omega + 0 + 0 + 0 && \text{as } \iota_Y\omega,\ \iota_X\omega,\ \omega \text{ are closed} \\
&= \mathrm{d}(\iota_Y \omega(X)) = \mathrm{d}(-\,\mathrm{d}g(X)) = -\,\mathrm{d}(X(g)) \\
&= -\,\mathrm{d}\{f, g\}
\end{aligned}$$

which proves the proposition. □

Then it is not difficult to show that the Poisson bracket gives a Lie algebra structure to $\mathrm{C}^\infty(M)$.

Corollary 2.5. *The vector space of smooth functions over M with the Poisson bracket forms a Lie algebra.*

Corollary 2.6. *There is a Lie algebra morphism*

$$(\mathrm{C}^\infty(M), \{\cdot,\cdot\}) \to (\mathfrak{X}(M)), [\cdot,\cdot]).$$

Let us return to Hamiltonian vector fields themselves.

Proposition 2.7. *Let X be a vector field on M with Hamiltonian f. Then f and ω are preserved under the flow of X.*

Proof. For f we have

$$Xf = \mathcal{L}_X f = \iota_X \,\mathrm{d}f + \mathrm{d}\iota_X f = -\iota_X \iota_X \omega + \mathrm{d}0 = \omega(X, X) = 0$$

as ω is skewsymmetric. For ω we have

$$\mathcal{L}_X \omega = \iota_X \,\mathrm{d}\omega + \mathrm{d}\iota_X \omega = \iota_X 0 - \mathrm{d}\mathrm{d}f = 0$$

as ω is closed. $\square$

The fact that ω is preserved under the flow of X seems to imply that X captures a 'symmetry' of our symplectic manifold. Let us make this more precise. Suppose X is a complete vector field, i.e. it has a flow $\gamma : \mathbb{R} \times M \to M$ such that for all $t \in \mathbb{R}$ the derivative of $\gamma_t := \gamma(t, \cdot)$ is X. Then for each t, γ_t is a diffeomorphism of M. We shall say γ is a symmetry of (M, ω) if for each t, γ_t is in fact a symplectomorphism, i.e. ω is invariant under γ_t.

Proposition 2.8. *For X a complete vector field with flow γ, $\mathcal{L}_X \omega = 0$ iff γ is a symmetry of (M, ω).*

Proof. Suppose $\mathcal{L}_X \omega = 0$. Let $x \in M$ and $a \in \mathbb{R}$. Then we have

$$\begin{aligned}
0 &= \mathrm{d}(\gamma_a)^*_x((\mathcal{L}_X \omega)_{\gamma_a(x)}) \\
&= \mathrm{d}(\gamma_a)^*_x\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \mathrm{d}(\gamma_s)^*_{\gamma_a(x)}(\omega_{\gamma_s(\gamma_a(x))})\Big) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \mathrm{d}(\gamma_{s+a})^*_x(\omega_{\gamma_{s+a}(x)}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=a} \mathrm{d}(\gamma_t)^*_x(\omega_{\gamma_t(x)}) \quad . \qquad \text{change of variables } t = s + a
\end{aligned}$$

Then by looking at local coordinates around x we have

$$\mathrm{d}(\gamma_a)^*_x(\omega_{\gamma_a(x)}) = \omega_x$$

thus $\gamma_a^* \omega = \omega$ for all $a \in \mathbb{R}$, hence ω is invariant under γ. The other direction follows immediately from the definition of $\mathcal{L}_X$. $\square$

Thus a (global) hamiltonian flow is a symmetry of our symplectic manifold. Note however that this only uses the fact that $\mathcal{L}_X \omega = 0$, that is that $\iota_X \omega$ is closed. This motivates the next definition.

Definition 2.9. A vector field X is locally Hamiltonian if $\iota_X \omega$ is closed. We define the vector space of such

The name locally Hamiltonian arises from the fact that if $\iota_X\omega$ is closed, then by the Poincaré Lemma it is locally 'integrable': for every $x \in M$ there exists an open neighbourhood U of x and a smooth function f on U such that f is a Hamiltonian for $X|_U$. The natural question then is when is every locally Hamiltonian vector field (globally) Hamiltonian? By definition X is a (globally) Hamiltonian vector field precisely when $\iota_X\omega$ is exact. By the isomorphism $\mathfrak{X}(M) \to \mathrm{T}^*(M)$ induced by ω, there is a surjection from $\mathcal{H}_{\mathrm{loc}}$ to $\mathrm{H}^1_{dR}(M)$ and so we have a short exact sequence

$$0 \longrightarrow \mathcal{H}(M) \longrightarrow \mathcal{H}_{\mathrm{loc}} \longrightarrow \mathrm{H}^1_{dR}(M) \longrightarrow 0$$

thus the obstruction is measured by $\mathrm{H}^1_{dR}(M)$.

Each complete local Hamiltonian vector field X defines a lie group action from $\mathbb{R}$ to M that acts symplectically — a symplectic lie group action on M. Suppose that in addition X is globally Hamiltonian, with Hamiltonian f. Then the level set of each regular point of f is a submanifold M and as we have that as f is preserved by the flow of X, X is tangent to the level sets of f. Thus each integral curve of X is a submanifold of a level set of f. So in some sense, f parametrises the integral curves of X.

2.2 Hamiltonian Group Actions

Let G be a lie group acting on M with smooth action $\lambda : G \times M \to M$. Then we say G acts symplectically if for each $g \in G$, $\lambda_g := \lambda(g, \cdot)$ is a symplectomorphism, i.e. $\lambda_g^*\omega = \omega$. Then as usual with Lie groups we can capture much of this action by passing to the lie group.

Let G be a lie group acting symplectically on M. Let $X \in \mathfrak{g}$, then its fundamental vector field $\underline{X}$ has flow $\lambda_{\exp tX}$. Suppose further that $\underline{X}$ is complete. Then by Proposition 2.8 we have that $\underline{X}$ is locally Hamiltonian, an infinitesimal symmetry of ω. This is a local notion however, and we would like the symmetry that G describes to be stronger. We would like $\underline{X}$ to be globally Hamiltonian for each $X \in \mathfrak{g}$. Thus we need a map $\tilde{\mu} : \mathfrak{g} \to \mathrm{C}^\infty(M)$ such that the diagram

$$\begin{tikzcd} & \mathrm{C}^\infty(M) \arrow[d] & \mathfrak{g} \arrow[l, "\tilde{\mu}"'] \arrow[d] & & \\ 0 \arrow[r] & \mathcal{H}(M) \arrow[r] & \mathcal{H}_{\mathrm{loc}}(M) \arrow[r] & \mathrm{H}^1_{dR}(M) \arrow[r] & 0 \end{tikzcd}$$

commutes. We also require $\tilde{\mu}$ to be a Lie algebra morphism. We call $\tilde{\mu}$ a comoment map and if for G there exists such a comoment map we call the action Hamiltonian. We denote the image of X under $\tilde{mu}$ by $\tilde{mu}_X$. We also require the 'dual' notion to the comoment map.

Proposition 2.10. *A Lie group G has a Hamiltonian action on a symplectic manifold (M, ω) if there exists a map $\mu : M \to \mathfrak{g}^*$ such that*

- *For each $X \in \mathfrak{g}$ the map*

$$p \mapsto \langle \mu(p), X \rangle = \tilde{\mu}_X(p),$$

- *μ is equivariant with respect to the action of G on M and the coadjoint action Ad^* on $\mathfrak{g}^*$.*

Proof. The proof that the second item shows $\tilde{\mu}$ is a Lie algebra morphism follows from [1] proposition III.1.3, but will only be studying abelian actions. □

Proposition 2.11. *Suppose that G acts on a symplectic manifold (M, ω) with moment map $\mu \colon M \to \mathfrak{g}$ and suppose H is a Lie subgroup of G with inclusion map τ. Then H has a Hamiltonian action on M with moment map $\nu := t^* \circ \mu \colon M \to \mathfrak{h}^*$ where t is the Lie algebra homomorphism induced by τ.*

Proof. The action of H on M is restriction of the action of G. Then for $X \in \mathfrak{h}$ its fundamental vector field is $\underline{t(X)}$. Let p be a point in M. Then

$$\tilde{\nu}_X(p) := \langle \nu(p), X \rangle = \langle t^* \circ \mu(p), X \rangle = \langle \mu(p), t(X) \rangle = \tilde{\mu}_{t(X)}(p)$$

and $\tilde{\mu}_{t(X)}$ is the Hamiltonian for $\underline{t(X)}$. so we only need to show H-equivariance. We have $t^* \circ \mathrm{Ad}^*_{t(X)} |_H = \mathrm{Ad}^*_X \circ t^*$ thus G-equivariance implies H-equivariance. □

Example 2.12. Define the function $f_j : \mathbb{C}^n \to \mathbb{R}$ by $f(z) = \frac{1}{2}\left|z_j\right|^2 + c_j$ where c_j is some real constant. Then in coordinates $(x_1, y_1, \ldots, x_n, y_n)$, $\mathrm{d}f_j = x_j \,\mathrm{d}x_j + y_j \,\mathrm{d}y_j$. We want to find its Hamiltonian vector field.
Write $X = \sum (X_k \partial/\partial x_k + Y_k \partial/\partial y_k)$. Then with $\omega = \sum \mathrm{d}x_k \wedge \mathrm{d}y_k$ we have that $\iota_X \omega = \sum(-Y_k \,\mathrm{d}x_k + X_k \,\mathrm{d}y_k)$. Thus $X_{f_j} = -y_j \partial/\partial x_j + x_j \partial/\partial y_j$ is the Hamiltonian vector field for f. Thus the flow $(t, z) \mapsto (\ldots, e^t z_j, \ldots)$ of X is a Hamiltonian $\mathbb{R}$-action with moment map f_j.

Example 2.13. Consider $\mathbb{T}^n$ acting on $\mathbb{C}^n$ by

$$(u_1, \ldots, u_j, \ldots, u_n) \cdot (z_1, \ldots, z_j, \ldots, z_n) = (u_1 z_1, \ldots, u_j z_j, \ldots, u_n z_n).$$

Then let $\xi_j = (\ldots, \theta_j, \ldots) \in \mathfrak{t}^n \cong \mathbb{R}^n$. Its flow is $(t, z) \mapsto (\ldots, e^t z_j, \ldots)$, thus its fundamental vector field is $X_j = -y_j \partial/\partial x_j + x_j \partial/\partial y_j$ with coordinates as in example 2.12. So, its vector field is Hamiltonian with Hamiltonian f_j. Now define the map $\tilde{\mu} : \mathbb{C}^n \to \mathfrak{t}^{n*} \cong \mathbb{R}^n$ by

$$\mu(z) = (f_1, \ldots, f_n)(z) = 1/2(|z_1|^2, \ldots, |z_n|^2) + (c_1, \ldots, c_n) \quad .$$

However $f_1, \ldots f_n$ Poisson commute i.e. their pairwise Poisson brackets are zero and as $\mathbb{T}^n$ is abelian this is sufficient for $\tilde{\mu}$ to be a comoment map and thus μ a moment map. The existence of μ shows that every fundamental vector field associated with elements of $\mathfrak{t}^n$ is Hamiltonian and so by proposition 2.8,0 ω is preserved under their flows. As $\mathbb{T}^n$ is connected and compact its exponential map $\exp : \mathfrak{t}^n \to \mathbb{T}^n$ is surjective thus *a fortiori* the Hamiltonian $\mathbb{T}^n$ action on $\mathbb{C}^n$ is symplectic.

Non-Example 2.14. Consider $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ acting on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ by addition on the first factor. Then $\mathbb{T}^2$ has a translation invariant symplectic form $\mathrm{d}x \wedge \mathrm{d}y$ so this action is symplectic. Let $1 \in \mathfrak{s}^1$, then its fundamental vector field on $\mathbb{T}^2$ is $\partial/\partial x$. Computing $-\iota_{\partial/\partial x}\omega$ we get $-\mathrm{d}y$ but this is precisely the element $(0, -1) \in \mathbb{R}^2 \cong \mathrm{H}^1_{dR}(\mathbb{T}^2)$ so there is no primitive for it. Thus the action is not Hamiltonian.

2.3 Symplectic Reduction

Example 2.15. Consider $\mathbb{S}^1$ acting on $\mathbb{C}^n$ by

$$u \cdot (z_1, \ldots, z_j, \ldots, z_n) = (uz_1, \ldots, uz_j, \ldots, uz_n)$$

where $n \geq 2$. Then for $1 \in \mathfrak{s}^1$ we have the fundamental vector field as

$$\underline{1} = \sum (-y_j \partial/\partial x_j + x_j \partial/\partial y_j)$$

and thus $\iota_{\underline{1}}\omega = -\sum (x_j \, \mathrm{d}x_j + y_j \, \mathrm{d}y_j)$. Hence we the action is Hamiltonian with moment map

$$\mu(z_1, \ldots, z_j, \ldots, z_n) = \frac{1}{2} \sum |z_j|^2 - c$$

with $c \in \mathbb{R}$ a constant. Then $\mathrm{d}\mu_z = [x_1, y_1, \ldots, x_j, y_j, \ldots, x_n, y_n]$ thus any $z \neq 0$ is a regular point for μ. Then $\mu^{-1}(0) = \{z \in \mathbb{C}^n : \sum |z_j|^2 = 2c\}$ so for $c > 0$ we have that $\mu^{-1}(0)$ is non-empty, and as $\mu^{-1}(0) \subset \mathbb{C}^n \setminus \{0\}$ we have that 0 is a regular value for μ, hence $\mu^{-1}(0)$ is a closed submanifold of $\mathbb{C}^n$, topologically $\mu^{-1}(0) \cong \mathbb{S}^{2n-1}$. Then $\mu^{-1}(0)$ is odd-dimensional so it cannot have a symplectic form.

However, as we will now see, $\mu^{-1}(0)$ is a coisotropic submanifold of $\mathbb{C}^n$. As $\iota_{\underline{1}}\omega_z = -\,\mathrm{d}\mu_z$ we have

$$(\mathrm{T}_z\,\mathbb{S}^1 \cdot z)^{\omega_z} = \ker \iota_{\underline{1}}\omega_z = \ker \mathrm{d}\mu_z = \mathrm{T}_z\,\mu^{-1}(0)$$

where $z \in \mu^{-1}(0)$ and $\mathbb{S}^1 \cdot z$ is the orbit of z. But μ is $\mathbb{S}^1$-equivariant hence $\mathbb{S}^1 \cdot z \subsetneq \mu^{-1}(0)$ thus $\mu^{-1}(0)$ is coisotropic. Let $i : \mu^{-1}(0) \to \mathbb{C}^n$ be the inclusion map. While we do not have a symplectic form on $\mu^{-1}(0)$, the pullback of ω via i gives us a closed 2-form on $\mu^{-1}(0)$ and the 'degenerate parts' are along the $\mathbb{S}^1$-trajectories. Importantly we have that $\mathbb{S}^1$ acts freely on $\mu^{-1}(0)$ so we can remove these 'degenerate parts' by forming the orbit space $\mu^{-1}(0)/\mathbb{S}^1$ with quotient map $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/\mathbb{S}^1$.

Theorem 2.16. *Let G be a compact Lie group acting freely on a manifold M. Then M/G is a manifold and $M \to M/G$ is a principal G-bundle.*

Proof. See [2] theorem 23.4. □

Theorem 2.17. *Let G be a compact Lie group with a Hamiltonian action on a symplectic manifold (M, ω) with moment map $\mu : M \to \mathfrak{g}^*$. Suppose that G acts freely on $\mu^{-1}(0)$. Then $\mu^{-1}(0)/G$ is a manifold. Let τ be the inclusion of $\mu^{-1}(0)$ and π the projection from $\mu^{-1}(0)$ to $\mu^{-1}(0)/G$. Then there exists a unique symplectic form ω_{red} on $\mu^{-1}(0)/G$ such that*

$$\tau^*\omega = \pi^*\omega_{red}.$$

Proof. In order to apply theorem 2.16, we need to show that $\mu^{-1}(0)$ is indeed a manifold. To this end, it will suffice[4] to show that 0 is a regular value for μ, that is for every $p \in \mu^{-1}(0)$ the linear map $\mathrm{d}\mu_p$ is surjective.

We claim that $\mathrm{im}\,(\mathrm{d}\mu_p) \subseteq \mathfrak{g}^*$ is the annihilator $\mathfrak{g}_p^0$ of the Lie algebra of the stabiliser of p. We have

$$\begin{aligned}
X \in \mathfrak{g}_p &\iff \underline{X}_p = 0 \\
&\iff \omega_p(\underline{X}_p, v) = 0 \text{ for all } v \in \mathrm{T}_p\,M \\
&\iff \iota_{X_p}\omega_p(v) = 0 \text{ for all } v \in \mathrm{T}_p\,M \\
&\iff -\,\mathrm{d}\tilde{\mu}_X|_p(v) = 0 \text{ for all } v \in \mathrm{T}_p\,M \\
&\iff \langle \mathrm{d}\mu_p(v), X\rangle = 0 \text{ for all } v \in \mathrm{T}_p\,M \\
&\iff X \in (\mathrm{im}(\mathrm{d}\mu_p))^0
\end{aligned}$$

[4] See [11] corollary 5.14.

Thus $\mathfrak{g}_p = (\mathrm{im}(\mathrm{d}\mu_p))^0$ and hence $\mathfrak{g}_p^0 = \mathrm{im}(\mathrm{d}\mu_p)$. But G acts freely[5] at p thus $\mathfrak{g}_p = 0$ and so $\mathrm{d}\mu_p$ is surjective hence 0 is a regular value for μ, and so $\mu^{-1}(0)$ is an embedded submanifold. Applying theorem 2.16, we have a submersion

$$\pi \colon \mu^{-1}(0) \to \mu^{-1}(0)/G.$$

The dimension of $\mu^{-1}(0)$ is $\dim M - \dim \ker \mathrm{d}\mu_p = \dim M - \dim G$ thus the dimension of $\mu^{-1}(0)/G$ is $\dim M - 2\dim G$.

By the first isomorphism theorem, the tangent space at a point $\pi(p) \in \mu^{-1}(0)/G$ is canonically isomorphic to $\ker(\mathrm{d}\mu_p)/\ker(\mathrm{d}\pi_p)$. But $\ker(\mathrm{d}\pi_p)$ is precisely the tangent space at p of the orbit $G \cdot p$. By proposition 1.11,

$$\mathrm{T}_p(G \cdot p) \cong \mathfrak{g}/\mathfrak{g}_p = \mathfrak{g}.$$

Thus if we show that the symplectic complement of $\ker(\mathrm{d}\mu_p)$ is $\mathfrak{g}$ by then by lemma 1.8 the restriction of ω to $\ker(\mathrm{d}\mu_p)$ descends to a symplectic form on $\mathrm{T}_{\pi(p)}(\mu^{-1}(0)/G)$. Fix $v \in \ker(\mathrm{d}\mu_p)$. Then for any $X \in \mathfrak{g}$

$$\omega_p(\underline{X_p}, v) = \langle \mathrm{d}\mu_p(v), X \rangle = 0.$$

Hence $\mathfrak{g} \subseteq (\ker(\mathrm{d}\mu_p))^\perp$. By lemma 1.7 we have

$$\dim \ker(\mathrm{d}\mu_p)^\perp = \dim \mathrm{T}_p M - \dim \ker(\mathrm{d}\mu_p) = \dim M - (\dim M - \dim G) = \dim G$$

hence we have $\mathfrak{g} = \dim \ker(\mathrm{d}\mu_p)^\perp$. Thus there exists a non-degenerate 2-form ω_{red} on $\mu^{-1}(0)/G$ such that $\pi^*\omega_{\mathrm{red}} = \tau^*\omega$. We have that $\tau^*\omega = \pi^*\omega_{\mathrm{red}}$ is smooth and thus by definition of the charts of $\mu^{-1}(0)/G$ (see [11]) ω_{red} is smooth. The pullback commutes with the exterior derivative thus

$$\pi^* \mathrm{d}\omega_{\mathrm{red}} = \mathrm{d}\pi^*\omega_{\mathrm{red}} = \mathrm{d}\tau^*\omega = \tau^*\mathrm{d}\omega = 0.$$

But $\mathrm{d}\pi_p$ at every p is surjective thus its dual is injective hence ω_{red} is closed. Suppose ω' is another 2-form such that $\pi^*\omega' = \tau^*\omega$. Then $\pi^*(\omega_{\mathrm{red}} - \omega') = 0$ thus by the reasoning before $\omega_{\mathrm{red}} = \omega'$. □

3 Delzant Spaces

We now will explore a family of 'nice' symplectic manifolds that have a combinatorial description.

[5] We do not need G to act freely on $\mu^{-1}(0)$ to show it is a submanifold, only locally free, i.e. G_p finite. However for the quotient we do need the free action for it to be a manifold.

Definition 3.1. A toric manifold is a compact connected symplectic manifold (M, ω) of dimension $2n$ such that it has an effective Hamiltonian action of the torus $\mathbb{T}^n$.

Definition 3.2 (Delzant Polytope)**.** A convex polytope $\Delta \subset \mathbb{R}^n$ is defined as the convex hull of finitely many points in $\mathbb{R}^n$. Let p be a vertex of Δ. We say p is simple if there are precisely n edges $v_1, \ldots, v_n$ of Δ meeting at p. We then say p is rational if for each jth edge v_j meeting p, there exists some $r_j \in \mathbb{Z}^n$ such that v_j lies on a ray of the form $p + tr_j$ for $t \geq 0$. Finally we say that p is smooth if these $r_1, \ldots, r_n$ can be chosen such that they form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$. Then we define a Delzant polytope as a convex polytope Δ such that each vertex p of Δ is simple, rational, and smooth.

In practice however, it is more useful to consider a Delzant polytope Δ as a subset of $(\mathbb{R}^n)^*$. As before, for each vertex of $p \in \Delta \subset (\mathbb{R}^n)^*$ we have a set $J_p := \{r_1, \ldots, r_n\} \subset (\mathbb{Z}^n)^*$ such that they form a $\mathbb{Z}$-basis of $(\mathbb{Z}^n)^*$. For each $r_{j_1}, \ldots, r_{j_{n-1}} \in J_p$ we define $u_{j_n} \in \mathbb{Z}^n \subset \mathbb{R}^n$ such that $\langle u_{j_n}, u_{j_k} \rangle = 0$ for $k = 1, \ldots, n-1$. Then there exists a scalar $\lambda_{j_n} \in \mathbb{R}$ such that the face of Δ spanned by $r_{j_1}, \ldots, r_{j_{n-1}}$ is given by the equation $\langle u_{j_n}, x \rangle = \lambda_{j_n}$. Then we also impose that u_{j_n} is primitive in the lattice $\mathbb{Z}^n$ and is 'inward pointing' in the sense that Δ is contained in the half-space given by $\langle u_{j_n}, x \rangle \geq \lambda_{j_n}$. One can also see that $u_1, \ldots, u_n$ then form a $\mathbb{Z}$ basis of $\mathbb{Z}^n$. Then we have described Δ as an intersection of half space of the form

$$\langle u_j, x \rangle \geq \lambda_j$$

for $j = 1, \ldots, d$.

3.1 Delzant Construction

Theorem 3.3 (elaborated from construction given in [6])**.** *If $\Delta \subset \mathbb{R}^n$ is a Delzant polytope, then there exists a toric manifold M_Δ such that Δ is the image of its moment map.*

Proof. Define the linear map $\pi \colon \mathbb{R}^d \to \mathbb{R}^n$ by $e_j \mapsto u_j$ where e_j is the jth standard basis vector of $\mathbb{R}^d$. Set $m = d - n$. As π is surjective, we have a short exact sequence

$$0 \longrightarrow \mathbb{R}^m \xrightarrow{\iota} \mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^n \longrightarrow 0 \ .$$

Let P be the matrix for π in the standard bases. Now as both e_j and u_j have integral coefficients, P is an integer matrix. Then in column echelon form, $P = QE$ where Q and E are rational matrices and E is a basis change of $\mathbb{R}^d$.

Then as Q is the reduced column echelon form of P it has precisely m zero columns. Then let J be the d by m matrix sending the basis element e_k of $\mathbb{R}^m$ to the basis element of $\mathbb{R}^d$ that represents the kth zero column of Q. Then J is an integer matrix and is an isomorphism from $\mathbb{R}^m$ to $\ker Q \leq \mathbb{R}^d$. Thus $E^{-1}J$ is a rational matrix and is an isomorphism from $\mathbb{R}^m$ to $\ker P \leq \mathbb{R}^d$. Hence we can scale $E^{-1}J$ by an integer to make it an integer matrix with image still $\ker P$. Therefore we can choose ι such that ι sends integer vectors to integer vectors. Thus we have induced maps

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^m \arrow[r, "\iota"] \arrow[d] & \mathbb{R}^d \arrow[r, "\pi"] \arrow[d] & \mathbb{R}^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{R}^m/\mathbb{Z}^m \arrow[r, "\tilde{\iota}"] \arrow[d, equal] & \mathbb{R}^d/\mathbb{Z}^d \arrow[r, "\tilde{\pi}"] \arrow[d, equal] & \mathbb{R}^n/\mathbb{Z}^n \arrow[r] \arrow[d, equal] & 0 \\
 & \mathbb{T}^m & \mathbb{T}^d & \mathbb{T}^n &
\end{tikzcd} \qquad (\star)$$

where the horizontal rows are exact. As we are dealing with finite dimensional vector spaces the dual sequence

$$0 \longrightarrow (\mathbb{R}^n)^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{\iota^*} (\mathbb{R}^m)^* \longrightarrow 0$$

is also exact. Then $\mathbb{T}^d$ acts on $\mathbb{C}^n$ Hamiltonially by

$$(t_1, \ldots, t_d) \cdot (z_1, \ldots, z_d) = (e^{2\pi i t_1} z_1, \ldots, e^{2\pi i t_d} z_d)$$

with moment map

$$\begin{aligned}
\psi \colon \mathbb{C}^d &\to (\mathbb{R}^d)^* \\
(z_1, \ldots, z_n) &\mapsto \frac{1}{2}(|z_1|^2, \ldots, |z_d|^2) + (c_1, \ldots, c_d)
\end{aligned}$$

where $(c_1, \ldots, c_d) \in (\mathbb{R}^d)^*$ is a constant. We set $c_j = \lambda_j$. Then $\mathbb{R}^m$ has an induced action by its inclusion i and by proposition . 2.11 its moment map is $\iota^* \circ \psi$. Then we define $\mathcal{Z} = (\iota^* \circ \psi)^{-1}(0)$ to be the level set at 0.

To prove any topological properties of $\mathcal{Z}$ we will need to show that ψ is proper, i.e. the preimage of a compact set under ψ is compact. Suppose A is a compact subset of $(\mathbb{R}^d)^*$. Then by Heine-Borel, A is closed and bounded. Then as ψ is continuous $\psi^{-1}(A)$ is closed leaving only to show it is bounded. As A is bounded there exists a positive $a \in \mathbb{R}$ such that if $y \in A$ then its jth component y_j has absolute value less than a. Suppose $y = \psi(z)$. Then $|z_j|^2 + \lambda_j < a$ hence $\psi^{-1}(A)$ is bounded and hence compact. Thus ψ is proper.

Next we want to deduce that ψ is closed. Suppose A is a closed subset of $\mathbb{C}^d$, and suppose $(x_n)_{n\geq 0}$ is a sequence in A such that $(\psi(x_n))_{n\geq 0}$ converges to some y in $(\mathbb{R}^d)^*$. Then $(\psi(x_n))_{n\geq 0} \cup \{y\}$ is compact thus its preimage under ψ is compact. However this preimage contains $(x_n)_{n\geq 0}$ and by (sequential) compactness there is a subsequence (x_{n_k}) that converges to some x which is in A, as A is closed. Finally, by continuity of ψ we have that $\psi(x) = y$. Thus we can deduce that $\psi(A)$ is closed, so ψ is closed.

Now we wish to show $\mathcal{Z}$ is connected. Certainly $\ker \iota^*$ is connected. Suppose however that $\mathcal{Z}$ was not connected, so was the disjoint union of two closed sets A and B. However, each fibre of ψ is connected (being a product of circles and points) thus $\ker \iota^*$ is the disjoint union of $\psi(A)$ and $\psi(B)$ which are closed, contradiction. Hence $\mathcal{Z}$ is connected.

Next we claim $\mathcal{Z}$ is compact. Firstly we observe that

$$y \in \operatorname{im}\psi \iff \langle e_j, y\rangle \geq \lambda_j \quad \forall j = 1, \ldots, d.$$

This is because if $y = \psi(z)$ then the jth component of y is $|z_j|^2 + \lambda_j$ which is greater or equal to λ_j — on the other hand, if the jth component y_j of y is greater or equal to λ_j for all j then $\psi(\sqrt{y_1 - \lambda_1}, \ldots, \sqrt{y_d - \lambda_d}) = y$. Now consider $\operatorname{im}\pi^* \cap \operatorname{im}\psi$. Then

$$\begin{aligned}
y \in \operatorname{im}\pi^* \cap \operatorname{im}\psi \iff\ & y = \pi^*(x) \text{ for some } x \in (\mathbb{R}^n)^* \\
& \quad \text{and } \langle e_j, y\rangle \geq \lambda_j \quad \forall j = 1, \ldots, d \\
\iff\ & y = \pi^*(x) \text{ and } \langle e_j, \pi^*(x)\rangle \geq \lambda_j \quad \forall j = 1, \ldots, d \\
\iff\ & y = \pi^*(x) \text{ and } \langle \pi(e_j), x\rangle \geq \lambda_j \quad \forall j = 1, \ldots, d \\
\iff\ & y = \pi^*(x) \text{ and } \langle u_j, x\rangle \geq \lambda_j \quad \forall j = 1, \ldots, d \\
\iff\ & y = \pi^*(x) \text{ and } x \in \Delta \iff y \in \pi^*(\Delta)
\end{aligned}$$

thus $\operatorname{im}\pi^* \cap \operatorname{im}\psi = \pi^*(\Delta)$. However, by exactness of the sequence we have that $\operatorname{im}\pi^* = \ker\iota^*$. Hence $\ker\iota^* \cap \operatorname{im}\psi = \pi^*(\Delta)$. But $\ker\iota^* \cap \operatorname{im}\psi$ is precisely $\psi(\mathcal{Z})$, thus $\psi(\mathcal{Z}) = \pi^*(\Delta)$. We know$\Delta$ is compact, so $\pi^*(\Delta)$ is compact. Consequently, as ψ is proper, $\mathcal{Z}$ is compact. Therefore we have shown that $\mathcal{Z}$ is connected and compact and so a quotient space of $\mathcal{Z}$ will be connected and compact.

The next step is to show that $\mathbb{T}^m$ acts freely on $\mathcal{Z}$. Suppose $\mathcal{F}$ is a face of Δ of codimension k. Then there exists a set $I_{\mathcal{F}} = \{j_1, \ldots, j_k\} \subseteq \{1, \ldots, d\}$ such that $p \in \mathcal{F}$ if and only if

$$\begin{aligned}
\langle u_{j_r}, p\rangle = \lambda_{j_r} \quad \forall j_r \in I_{\mathcal{F}} \iff\ & \langle \pi(e_{j_r}), p\rangle = \lambda_{j_r} \quad \forall j_r \in I_{\mathcal{F}} \\
\iff\ & \langle e_{j_r}, \pi^*(p)\rangle = \lambda_{j_r} \quad \forall j_r \in I_{\mathcal{F}}.
\end{aligned}$$

Hence for $z \in \mathcal{Z}$

$$\begin{aligned}
\psi(z) \in \pi^*(\mathcal{F}) \iff\ & |z_{j_r}|^2 + \lambda_{j_r} = \lambda_{j_r} \quad \forall j_r \in I_{\mathcal{F}} \\
\iff\ & z_{j_r} = 0 \quad \forall j_r \in I_{\mathcal{F}}.
\end{aligned}$$

Then define the sets

$$\mathcal{Z}|_{\mathcal{F}} = \{z \in \mathcal{Z} : z_{j_r} = 0 \text{ if } j_r \in I_{\mathcal{F}}\}$$

and

$$\mathbb{T}^d|_{\mathcal{F}} = \{(x_1, \ldots, x_d) + \mathbb{Z}^d \in \mathbb{T}^d : x_l = 0 \text{ if } l \notin I_{\mathcal{F}}\}.$$

Then $\mathbb{T}^d|_{\mathcal{F}}$ is contained in the stabliser of $z \in \mathcal{Z}|_{\mathcal{F}}$ for the $\mathbb{T}^d$ action. If $\mathcal{F}$ is a face of $\mathcal{F}'$ then $\mathbb{T}^d|_{\mathcal{F}'} \subseteq \mathbb{T}^d|_{\mathcal{F}}$.

> **Aside**
>
> Define $\mathring{\mathcal{F}}$ to be the 'interior' of the face $\mathcal{F}$ i.e. the subset of $\mathcal{F}$ that is disjoint from any face of lower dimension. Then if $p \in \mathring{\mathcal{F}}$ we have that $\langle u_l, p\rangle > \lambda_l$ for all $l \in \{1, \ldots, d\} \setminus I_{\mathcal{F}}$ thus if $\psi(z) = \pi^*(p)$ then $z_j = 0$ if and only if $j \in I_{\mathcal{F}}$. Defining the set
>
> $$\mathcal{Z}|_{\mathring{\mathcal{F}}} = \{z \in \mathcal{Z} : z_j = 0 \iff j \in I_{\mathcal{F}}\}$$
>
> then $\mathbb{T}^d|_{\mathcal{F}}$ *is* the stabiliser of $z \in \mathcal{Z}|_{\mathring{\mathcal{F}}}$ for the $\mathbb{T}^d$ action.

Now suppose $\mathcal{F}$ is a vertex. Then $|I_{\mathcal{F}}| = n$. Without out loss of generality, set $I_{\mathcal{F}} = \{1, \ldots, n\}$. For $z \in \mathcal{Z}_{\mathcal{F}}$, the stabiliser of z is

$$\mathbb{T}^d|_{\mathcal{F}} = \{(x_1, \ldots, x_n, 0, \ldots, 0\} + \mathbb{Z}^d \in \mathbb{T}^d \ : x_1, \ldots, x_n \in \mathbb{R}\}.$$

which is a subgroup of $\mathbb{T}^d$. By definition of a Delzant polytope, the $u_1, \ldots, u_n$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$. Hence $\hat{\pi}$ restricted to $\mathbb{T}^d_{\mathcal{F}}$ is a group isomorphism, in particular, a group monomorphism. Thus $\ker \tilde{\pi} \cap \mathbb{T}^d_{\mathcal{F}}$ is trivial. Exactness gives $\tilde{\iota}(\mathbb{T}^m) = \ker \tilde{\pi}$, therefore the stabiliser for each $z \in \mathcal{Z}_{\mathcal{F}}$ for the $\mathbb{T}^m$ action is trivial. Then for any face $\mathcal{F}'$ containing the vertex $\mathcal{F}$ the stabiliser for each point in $\mathcal{Z}_{\mathcal{F}'}$ for the $\mathbb{T}^m$ action is contained in the stabiliser for each point of $\mathcal{Z}_{\mathcal{F}}$ and so is trivial. Consequently, we have shown that $\mathbb{T}^m$ acts freely and Hamiltonially on $\mathcal{Z}$ thus by theorem 2.17 we can reduce the space to form the symplectic manifold M_Δ with reduced symplectic form ω_Δ. The dimension of M_Δ is $2d - 2m = 2n$.

Now we will show there is a Hamiltonian $\mathbb{T}^n$ action on M_Δ. Again, let $\mathcal{F}$ be a vertex of Δ. Then as before, $\widetilde{\pi}$ restricted to $\mathbb{T}^d|_{\mathcal{F}}$ is an isomorphism. Thus it has an inverse $\sigma\colon \mathbb{T}^n \to \mathbb{T}^d$ so we have that σ is a right inverse for π. Thus we have that the sequence $(\star)$ splits so we have that $(\iota, \sigma)\colon \mathbb{T}^m \oplus \mathbb{T}^n \to \mathbb{T}^d$ is an isomorphism. Therefore $\mathbb{T}^n$ has an action on M_Δ. But is this action

Hamiltonian?

$$\begin{tikzcd}
 & & (\mathbb{R}^n)^* \arrow[d, "\pi^*"] & \\
\mathcal{Z} \arrow[r, hook, "\tau"] \arrow[d, two heads, "p"] & \mathbb{C}^d \arrow[r, "\psi"] & (\mathbb{R}^d)^* \arrow[r, "\sigma^*"] & (\mathbb{R}^n)^* \\
M_\Delta \arrow[urrr, dotted, "\mu_\Delta"] & & &
\end{tikzcd}$$

Consider the composition $\sigma^* \circ \psi \circ \tau$ where τ is the inclusion of $\mathcal{Z}$ into $\mathbb{C}^d$. Then as ψ is the moment map for $\mathbb{T}^d$, *a fortiori* it is equivariant under the $\mathbb{T}^m$ action, and as $\mathbb{T}^d$ is abelian, it is constant along $\mathbb{T}^m$ orbits. Thus the composition $\sigma^* \circ \psi \circ \tau$ descends to the quotient, giving us a unique map $\mu_\Delta \colon M_\Delta \to (\mathbb{R}^n)^*$ such that $\mu_\Delta \circ p = \sigma^* \circ \psi \circ \tau$. Then

$$\begin{aligned}
\operatorname{im} \mu_\Delta &= \operatorname{im} \mu_\Delta \circ p \\
&= \operatorname{im} \sigma^* \circ \psi \circ \tau \\
&= \sigma^*(\psi(\mathcal{Z})) \\
&= \sigma^*(\pi^*(\Delta)) \\
&= \mathrm{id}^*(\Delta) = \Delta.
\end{aligned}$$

We also have that the action of $\mathbb{T}^n$ on M_Δ is effective. Again by 2.11, the moment map for the $\mathbb{T}^n$ action is $\sigma^* \circ \psi \circ \tau$. Let $X \in \mathbb{R}^n$ and let fundamental vector field for the $\mathbb{T}^n$ action on $\mathcal{Z}$ be $\underline{\sigma(X)}$. Thus

$$p^*(\langle \mathrm{d}\mu_\Delta, X\rangle) = \langle \mathrm{d}(\mu_\Delta \circ p), X\rangle = -\iota_{\underline{\sigma(X)}} \tau^* \omega = -\iota_{\underline{\sigma(X)}} p^* \omega_\Delta$$

Then p is a submersion so its pullback is injective thus we have that μ_Δ is actually the moment map for the $\mathbb{T}^n$ action.

□

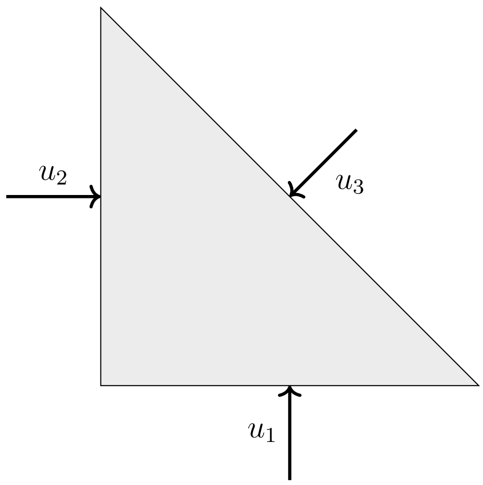
The symplectic form on $\mathbb{C}^d$ is Kähler (see [2]) and as the action of $\mathbb{T}^m$ preserves the complex structure of $\mathcal{Z}$, the reduced symplectic form is also Kähler.

Corollary 3.4. *$\mu_\Delta \colon M_\Delta \to \Delta$ maps the fixed points of the $\mathbb{T}^n$ action bijectively onto the vertices of Δ.*

Proof. Let $\mathcal{F}$ be a vertex of Δ. Then as in the proof of 3.3, there is a section $\sigma \colon \mathbb{T}^n \to \mathbb{T}^d$ thus

$$\mathbb{T}^d \cong \mathbb{T}^m \oplus \mathbb{T}^n. \tag{†}$$

Figure 1: Delzant polytope for $\mathbb{C}\mathrm{P}^2$



However we have also that for each $z \in \mathcal{Z}_{\mathring{\mathcal{F}}}$ $\mathbb{T}^n$ is the stabiliser of z for the $\mathbb{T}^d$ action in the decomposition (†). Notice however that $\mathbb{T}^d$ acts transitively on $\mathcal{Z}_{\mathring{\mathcal{F}}}$. Pick a $z \in \mathcal{Z}_{\mathring{\mathcal{F}}}$. Then we have

$$\mathcal{Z}_{\mathring{\mathcal{F}}} = \mathbb{T}^d \cdot z = (\mathbb{T}^m \oplus \mathbb{T}^n) \cdot z = \mathbb{T}^m \cdot z$$

hence $\mathbb{T}^m$ acts transitively on each $\mathcal{Z}_{\mathring{\mathcal{F}}}$ so each $\mathcal{Z}_{\mathring{\mathcal{F}}}$ is projected down to a single point on which $\mathbb{T}^n$ fixes. We also have that these are in bijection with the vertices of Δ. $\square$

Example 3.5. For $n \in \mathbb{N}_{\geq 0}$, define the n-simplex as the convex hull Δ^n of the points

$$0, e_1, \ldots, e_n$$

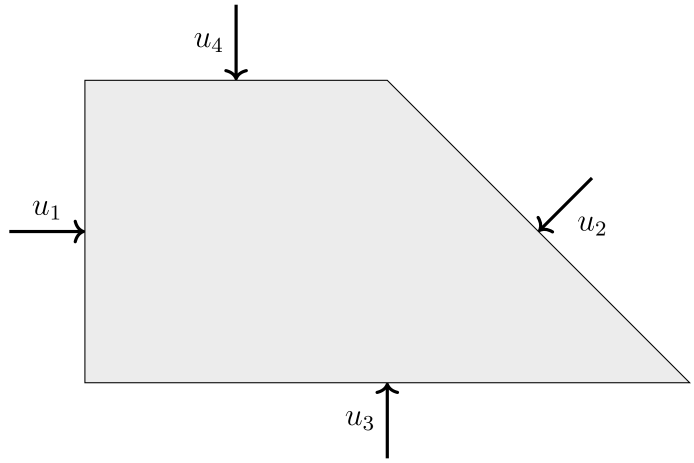
where e_i is the ith standard basis vector of $\mathbb{R}^n$. Δ^n is then an n-dimensional Delzant polytope. Described in terms of the dual space, we set the inward-pointing normal vectors as

$$u_i := e_i \text{ for } 1 \leq i \leq n$$

and

$$u_{n+1} := -\sum_{i=1}^{n} e_i$$

Figure 2: Delzant polytope for second Hirzebruch surface



with weights $\lambda_i = 0$ for $1 \leq i \leq n$ and $\lambda_{n+1} = -1$. In the standard bases our maps are

$$\iota = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad \pi = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \ddots & & 0 & -1 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & -1 \end{bmatrix}$$

with ι^* and π^* their transposes. Thus we have that

$$(\iota^* \circ \phi)(z) = \begin{bmatrix} 1 & \cdots & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2|z_1|^2 \\ \vdots \\ 1/2|z_n|^2 \\ 1/2|z_{n+1}|^2 - 1 \end{bmatrix} = \frac{1}{2} \sum_{j=1}^{n+1} |z_j|^2 - 1$$

Then just as in example 2.15 we have that M_{Δ^n} is diffeomorphic to $\mathbb{C}\mathrm{P}^n$.

Example 3.6. Define Δ_{H}^n as the convex polytope defined by the vertices $(0,0), (n+1,0), (1,1), (0,1) \in \mathbb{R}^2$. Then Δ_{H}^n is a Delzant polytope. In terms of the dual space, we have

$$\begin{array}{l|l} u_1 = (1,0) & \lambda_1 = 0 \\ u_2 = (-1,-n) & \lambda_2 = -n-1 \\ u_3 = (0,1) & \lambda_3 = 0 \\ u_4 = (0,-1) & \lambda_4 = -1 \end{array} .$$

Then in the standard basis our maps are

$$\iota = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ -n & 1 \end{bmatrix}, \qquad\qquad \pi = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -n & 1 & -1 \end{bmatrix}$$

with i^* and π^* their transposes. Then we have that

$$\begin{aligned}
(\iota^* \circ \psi^*)(z) &= \begin{bmatrix} 1 & 1 & 0 & -n \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2|z_1|^2 \\ 1/2|z_2|^2 - n - 1 \\ 1/2|z_3|^2 \\ 1/2|z_4|^2 - 1 \end{bmatrix} \\
&= \begin{bmatrix} 1/2|z_1|^2 + 1/2|z_1|^2 - n/2|z_4|^2 - 1 \\ 1/2|z_3|^2 + 1/2|z_4|^2 - 1 \end{bmatrix}
\end{aligned}$$

Thus our Delzant space is

$$M_{\Delta^n_{\mathrm{H}}} = \left\{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : {\scriptstyle \begin{matrix} |z_1|^2 + |z_2|^2 - n|z_4|^2 = 2 \\ |z_3|^2 + |z_4|^2 = 2 \end{matrix}} \right\} / \mathbb{T}^2$$

where $\mathbb{T}^2$ (now viewing it as a multiplicative group) acts on $(\iota^* \circ \psi^*)^{-1}(0)$ by

$$(t_1, t_2) \cdot (z_1, z_2, z_3, z_4) = (t_1 z_1, t_1 z_2, t_2 z_3, t_2 t_1^{-n} z_4).$$

It not immediately clear however what $M_{\Delta^n_{\mathrm{H}}}$ actually is — at least topologically.

3.2 Complex Construction

We want an analogous construction to Delzant's where we complexify the torus action, which will simplify the subset of $\mathbb{C}^d$ that we are quotienting by.

As before, the 'Delzant data' forms a short exact sequence

$$0 \longrightarrow \mathbb{R}^m \xrightarrow{\ \iota\ } \mathbb{R}^d \xrightarrow{\ \pi\ } \mathbb{R}^n \longrightarrow 0$$

where in the standard bases ι and π are integer matrices. Then we have the complexification

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}^m \arrow[r, "\iota_{\mathbb{C}}"] \arrow[d] & \mathbb{C}^d \arrow[r, "\pi_{\mathbb{C}}"] \arrow[d] & \mathbb{C}^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C}^m/\mathbb{Z}^m \arrow[r, "\widetilde{\iota}_{\mathbb{C}}"] & \mathbb{C}^d/\mathbb{Z}^d \arrow[r, "\widetilde{\pi}_{\mathbb{C}}"] & \mathbb{C}^n/\mathbb{Z}^n \arrow[r] & 0
\end{tikzcd}$$

which is still exact. We still have that $\iota_{\mathbb{C}}$ and $\pi_{\mathbb{C}}$ have integer matrices thus we can quotient to get a short exact sequence of complexified tori. In fact as $\iota_{\mathbb{C}}$ and $\pi_{\mathbb{C}}$ are integer matrices, as additive groups we can decompose

$$\mathbb{C}^m/\mathbb{Z}^m \cong \mathbb{T}^m \oplus i\mathbb{R}^m. \tag{$\star$}$$

Likewise for d and n. Define $\mathbb{T}^m_{\mathbb{C}} = \mathbb{C}^m/\mathbb{Z}^m$, similarly for d and n. Then $\mathbb{T}^d_{\mathbb{C}}$ has a proper action on $\mathbb{C}^d$ by

$$\begin{aligned}
&([x_1] + iy_1, \dots, [x_n] + iy_n) \cdot (z_1, \dots, z_n)\\
&= (e^{2\pi i(x_1+iy_1)} z_1, \dots, e^{2\pi i(x_n+iy_n)} z_n)\\
&= (e^{2\pi i x_1} e^{-2\pi y_1} z_1, \dots, e^{2\pi i x_n} e^{-2\pi y_n} z_n)
\end{aligned}$$

where $[x_j]$ is the conjugacy class of $x_j \in [0, 1)$. As in the proof of 3.3 we have that for each face $\mathcal{F}$ of Δ

$$\mathcal{Z}|_{\mathring{\mathcal{F}}} = \{z \in \mathcal{Z} : z_j = 0 \iff j \in I_{\mathcal{F}}\}$$

are the orbits of the $\mathbb{T}^d$ action. For the $\mathbb{T}^d_{\mathbb{C}}$ action, we define

$$\mathbb{C}^d|_{\mathring{\mathcal{F}}} = \{w \in \mathbb{C}^d : w_j = 0 \iff j \in I_{\mathcal{F}}\}.$$

which is a $\mathbb{T}^d_{\mathbb{C}}$ orbit. N.B. that the stabiliser for this action in terms of the decomposition ($\star$) is

$$\mathbb{T}^d|_{\mathcal{F}} \oplus i\mathbb{R}^d|_{\mathcal{F}}$$

where

$$\mathbb{R}^d|_{\mathcal{F}} = \{x \in \mathbb{R}^d : x_l = 0 \iff j \notin I_{\mathcal{F}}\}.$$

From the proof of 3.3 we know that $\tilde{\iota}_{\mathbb{C}}(\mathbb{T}^m) \cap \mathbb{T}^d|_{\mathcal{F}}$ is trivial. The same proof, mutatis mutandis, shows that $\iota_{\mathbb{C}}(i\mathbb{R}^m) \cap i\mathbb{R}^d|_{\mathcal{F}}$ is also trivial. Thus $\mathbb{T}^d_{\mathbb{C}}$ acts freely on $\mathbb{C}^d|_{\mathring{\mathcal{F}}}$. Collecting together all these orbits as

$$\mathbb{C}^d|_{\Delta} = \bigcup_{\mathcal{F} \text{ face of } \Delta} \mathbb{C}^d|_{\mathring{\mathcal{F}}}$$

we have that $\mathbb{T}^m_{\mathbb{C}}$ acts freely on $\mathbb{C}^d|_{\Delta}$.

Next, we want to show that $\mathbb{C}^d|_{\Delta}$ is an open subset of $\mathbb{C}^d$ and so is a complex manifold. Consider Δ, a face of itself. Then $I_{\Delta} = \varnothing$ hence $\mathbb{C}^d|_{\mathring{\Delta}} = (\mathbb{C}^{\times})^d$ which is open. Now suppose $w \in \mathbb{C}^d|_{\mathring{\mathcal{F}}}$. Then if $j \notin I_{\mathcal{F}}$, $w_j \neq 0$ so set B_{w_j} as some open ball in $\mathbb{C}^{\times}$ containing w_j. Then if $j \in I_{\mathcal{F}}$, $w_j = 0$ so set B_{w_j} as some open ball in $\mathbb{C}$ around 0. In each case then, we have that

$$\mathrm{B}_{w_j} \subseteq \mathrm{proj}_j(\mathbb{C}^d|_{\mathring{\Delta}} \cup \mathbb{C}^d|_{\mathring{\mathcal{F}}}) \subseteq \mathrm{proj}_j(\mathbb{C}^d|_{\Delta})$$

so that the cartesian product of the B_j's is an open subset contained in $\mathbb{C}^d|_{\Delta}$ around w. Hence $\mathbb{C}^d|_{\Delta}$ is open.

The next proposition shows that the two definitions of M_{Δ} are compatible.

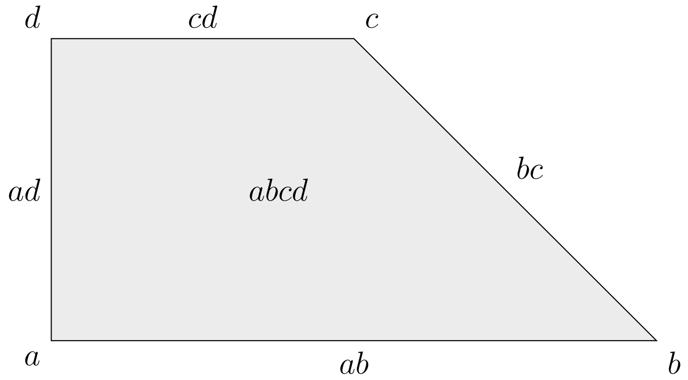
Proposition 3.7. *$\mathcal{Z} \subseteq \mathbb{C}^d|_\Delta$ and each $\mathbb{T}^m_{\mathbb{C}}$ orbit in $\mathbb{C}^d|_\Delta$ intersects $\mathcal{Z}$ in a $\mathbb{T}^m$ orbit.*

Proof. See [6] theorem 1.4 appendix 1. □

Example 3.8. For Δ^n the collection of $I_{\mathcal{F}}$ are all the *proper* subsets of $\{1, \dots, n, n+1\}$. Thus $\mathbb{C}^{n+1}|_{\Delta^n} = \mathbb{C}^{n+1} \setminus 0$, and $\mathbb{C}^\times$ acts on $\mathbb{C}^{n+1}|_{\Delta^n}$ by

$$\alpha \cdot (w_1, \dots, w_n, w_{n+1}) = (\alpha w_1, \dots, \alpha w_n, \alpha w_{n+1}).$$

Example 3.9. We return to example 3.6 and consider the complex construction. Label the faces of Δ^n_{H} as in the diagram below.



Then computing $\mathbb{C}^4|_{\mathring{\mathcal{F}}}$ for each face

face $\mathcal{F}$	$I_{\mathcal{F}}$	$\mathbb{C}^4|_{\mathring{\mathcal{F}}}$
a	{1,3}	$z_1, z_3 = 0$ and $z_2, z_4 \neq 0$
b	{2,3}	$z_2, z_3 = 0$ and $z_1, z_4 \neq 0$
c	{2,4}	$z_2, z_4 = 0$ and $z_1, z_3 \neq 0$
a	{1,4}	$z_1, z_4 = 0$ and $z_2, z_3 \neq 0$
ab	{3}	$z_3 = 0$ and $z_1, z_2, z_4 \neq 0$
bc	{2}	$z_2 = 0$ and $z_1, z_3, z_4 \neq 0$
cd	{4}	$z_4 = 0$ and $z_1, z_2, z_3 \neq 0$
ad	{1}	$z_1 = 0$ and $z_2, z_3, z_4 \neq 0$
$abcd$	∅	$z_1, z_2, z_3, z_4 \neq 0$

Thus

$$\mathbb{C}^4|_{\Delta^n_{\mathrm{H}}} = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : (z_1, z_2) \neq (0,0) \text{ and } (z_3, z_4) \neq (0,0)\}$$

where $(\mathbb{C}^\times)^2$ acts on $\mathbb{C}^4|_{\Delta^n_{\mathrm{H}}}$ by

$$(\alpha_1, \alpha_2) \cdot (z_1, z_2, z_3, z_4) = (\alpha_1 z_1, \alpha_1 z_1, \alpha_2 z_3, \alpha_2 \alpha_1^{-n} z_4).$$

It is much easier now to see what $M_{\Delta^n_{\mathrm{H}}}$ is, but first we will need a definition.

Define for $n \in \mathbb{Z}$

$$\mathbb{L}_{(-n)} := (\mathbb{C}^2 \setminus 0 \times \mathbb{C})/ \sim_{(-n)}$$

where

$$(u, v, w) \sim_{(-n)} (\alpha u, \alpha v, \alpha^{-n} w) \quad \alpha \in \mathbb{C}^\times.$$

Define $q\colon \mathbb{L}_{(-n)} \to \mathbb{C}\mathrm{P}^1 \cong \mathbb{S}^2$ by $q\colon [(u, v, w)] \mapsto [u, v]$. Then $\mathbb{L}_{(-n)}$ is a complex line bundle on $\mathbb{C}\mathrm{P}^1$. Some specific examples are

- $\mathbb{L}_{(0)}$ is the trivial line bundle.
- $\mathbb{L}_{(-1)}$ is the tautological line bundle.
- $\mathbb{L}_{(1)}$ is the line bundle associated to the Hopf fibration.

Consider then the complex vector bundle $\mathbb{L}_{(-n)} \oplus \underline{\mathbb{C}}$ where $\underline{\mathbb{C}}$ is the trivial line bundle on $\mathbb{C}\mathrm{P}^1$. Then we construct its associated projective bundle $\mathbb{P}(\mathbb{L}_{(-n)} \oplus \underline{\mathbb{C}})$ where the fibre above $[u, v]$ is

$$((\mathbb{L}_{(-n)}{}_{[u,v]} \oplus \mathbb{C}) \setminus 0)/ \sim$$

where $(w, z) \sim (\beta w, \beta z)$ for $\beta \in \mathbb{C}^\times$. Then we can see that

$$M_{\Delta^n_{\mathrm{H}}} = \mathbb{P}(\mathbb{L}_{(-n)} \oplus \underline{\mathbb{C}}) =: \mathrm{H}_n.$$

For $n \geq 0$, H_n is called the nth Hirzebruch surface. Topologically, they are a family of $\mathbb{S}^2$-bundles over $\mathbb{S}^2$. H_0 is just $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$. We have now shown that they are compact Kähler manifolds of dimension $2n$ with an effective Hamiltonian $\mathbb{T}^n$ action.

4 Geometric Quantisation

4.1 The Idea

Classical Mechanics:

In the Hamiltonian formulation of classical mechanics we view the state of some mechanical system as a point $p \in M$ on some smooth manifold M, endowed with a symplectic form ω. M is called the phase space of the system, the prototypical example being the cotangent bundle $\mathrm{T}^*(W)$ of some arbitrary smooth manifold W endowed with the canonical symplectic form as in example 1.9. Here W describes the 'position' or 'configuration' space of the system, while the tangent space describes the 'momenta space'. The physics of the system is defined by some smooth function H on M which describes some conserved quantity of the system, typically the total energy. The Hamiltonian vector field X_H associated to H then describes how the system behaves with respect to time. An observable of the system is a smooth function $f \in \mathrm{C}^\infty(M)$. The vector space $\mathrm{C}^\infty(M)$ comes with a Lie algebra structure, the Poisson bracket. In particular for an observable f, $\{f, H\}$ describes how the observable changes with the time evolution of the system — if $\{f, H\} = 0$ then f is called a conserved quantity of the system.

Quantum Mechanics:

We now give a brief description of what we define as a quantum system. The underlying set is a (separable) complex Hilbert space $\mathcal{H}$. States of the quantum system are 'rays' in $\mathcal{H}$ i.e. complex lines $\{\lambda v : \lambda \in \mathbb{C}\}$ with $v \in \mathcal{H}$. The quantum observables are self-adjoint operators mapping $\mathcal{H}$ into itself. Given two such observables A and B, we have that their commutator [A,B] is skew-adjoint, thus $i[A, B]$ is self adjoint. Therefore we can equip the quantum observables on $\mathcal{H}$ with a Lie algebra structure.

At this formal level, there seems to be some similarity between these two structures. The aim of quantisation is to formalise this similarity by a programme that takes some classical system and outputs a quantum one while preserving as much structure as possible. Geometric quantisation in particular attempts this by focusing on the geometry of the state space itself. Given a symplectic manifold (M, ω), we want to associate to it some separable complex Hilbert space $\mathcal{H}$ where we can send classical observables on M to quantum observables on $\mathcal{H}$ in some 'functorial' way. Explicitly we want a an injective (possibly partial) mapping $\mathcal{Q}$ from the smooth functions on M to the self-adjoint operators on $\mathcal{H}$ such that for smooth functions f and g

Q1 (linear) $\mathcal{Q}(\lambda f + \mu g) = \lambda \mathcal{Q}(f) + \mu \mathcal{Q}(g)$ for real scalars λ, μ,

Q2 (normalised) $\mathcal{Q}(\mathbf{1}) = \mathrm{id}_{\mathcal{H}}$, and

Q3 (Lie morphism) $[\mathcal{Q}(f), \mathcal{Q}(g)] = i\hbar\mathcal{Q}(\{f, g\})$ where $\hbar$ is the reduced Planck constant.

If we have found a $\mathcal{Q}$ and $\mathcal{H}$ satisfying **Q**1 to **Q**3 then we say we have prequantised the system (M, ω).

First Attempt

(following [4]).

Let (M,ω) be our symplectic manifold of dimension $2n$. The simplest route would be to consider $\mathrm{C}_c^\infty(M,\mathbb{C})$, the collection of smooth complex-valued functions on M with compact support. This is certainly a complex vector space with inner product

$$\langle \phi, \psi \rangle = \int_M \overline{\phi}\psi \frac{\omega^{\wedge n}}{n!}.$$

Define $\mathcal{C}$ as its completion as a Hilbert space. We already have ample linear operators on this space — vector fields on M are derivations of $\mathrm{C}^\infty(M)$ thus can be extended to linear operators on $\mathrm{C}_c^\infty(M,\mathbb{C})$ and hence to $\mathcal{C}$. For each function $f \in \mathrm{C}^\infty(M)$ we already know how to associate to it such an operator, its Hamiltonian vector field X_f Leaving for now the question of whether this operator is actually self-adjoint, could we define $\mathcal{G}$ by sending a function to its Hamiltonian vector field? Certainly $f \mapsto i\hbar X_f$ satisfies **Q**1 and **Q**3, however $X_{\mathbf{1}} = 0$ so it fails condition Q2.

We could try to 'bodge' this map by adding f itself. Set $\mathcal{G}\colon f \mapsto i\hbar X_f + f$ where f acts as linear operator by multiplication. This satisfies **Q**1 and **Q**2, however checking **Q**3 we see that for $\phi \in \mathcal{C}$

$$\begin{aligned}
&[\mathcal{G}(f), \mathcal{G}(g)](\phi) \\
&= [i\hbar X_f, i\hbar X_g](\phi) + [f, i\hbar X_g](\phi) + [i\hbar X_f, g](\phi) + [f,g](\phi) \\
&= -\hbar^2 X_{\{f,g\}}(\phi) + i\hbar(f X_g(\phi) - X_g(f\phi) + X_f(g\phi) - g X_f(\phi)) && \text{[a]} \\
&= -\hbar^2 X_{\{f,g\}}(\phi) \\
&\qquad + i\hbar(f X_g(\phi) - f X_g(\phi) - X_g(f)\phi + g X_f(\phi) + X_f(g)\phi - g X_f(\phi)) && \text{[b]} \\
&= -\hbar^2 X_{\{f,g\}}(\phi) + i\hbar(-\{g,f\}(\phi) + \{f,g\}(\phi)) \\
&= i\hbar\mathcal{G}(\{f,g\})(\phi) + i\hbar\{f,g\}(\phi).
\end{aligned}$$

Where line [a] follows from proposition 2.3 and as the commutator of functions is zero. Line [b] follows by the product rule.

So **Q**3 fails as we have an extra $\{f,g\}$ component. To make this easier for ourselves, assume for a moment that ω is exact with potential form α.

Then we have that[6]

$$
\begin{aligned}
&\{f, g\} \cdot \phi \\
&= \omega(X_f, X_g) \cdot \phi \\
&= \mathrm{d}\alpha(X_f, X_g) \cdot \phi \\
&= X_f(\alpha(X_g)) \cdot \phi - X_g(\alpha(X_f)) \cdot \phi - \alpha([X_f, X_g]) \cdot \phi \\
&= \big(X_f(\alpha(X_g) \cdot \phi) - \alpha(X_g) \cdot X_f(\phi)\big) \\
&\quad - \big(X_g(\alpha(X_f) \cdot \phi) - \alpha(X_f) \cdot X_g(\phi)\big) \\
&\quad - \alpha(X_{\{f,g\}}) \cdot \phi \\
&= [X_f, \alpha(X_g)](\phi)] + [X_g, \alpha(X_f)](\phi) - \alpha(X_{\{f,g\}}).
\end{aligned}
$$

Now the way is clear: if we set $\mathcal{G}\colon f \mapsto i\hbar X_f - \alpha(X_f) + f$ we have satisfied conditions **Q**1, **Q**2, and **Q**3. In fact, although we will not prove it just yet, $\mathcal{G}(f)$ is self-adjoint. However, this all relies on the fact that ω is exact. We want to be able quantise Delzant spaces — and compact manifolds cannot have an exact symplectic form by proposition 1.5.

Due to the Poincaré lemma we can still do this operation locally, on a contractible open subset of M. We could then try to 'glue' the 1-forms α together — which brings us to the concept of line bundles.

4.2 Complex Line Bundles, Connections, and Curvature

(adapted from [12]).

Complex Line Bundles

Definition 4.1. A triple $(\mathbf{L}, \pi, M)$ is a complex line bundle if

- $\mathbf{L}$ is a smooth manifold and π is a smooth surjective map.
- For every $x \in M$, the fibre $\mathbf{L}_x := \pi^{-1}(\{x\})$ has a 1-dimensional complex structure.
- For each $x \in M$ there is an open set $U \ni x$ of M and a diffeomorphism

$$\psi\colon U \times \mathbb{C} \to \pi^{-1}(U)$$

[6] here $\cdot$ is ordinary pointwise multiplication

such that

$$\begin{tikzcd} U \times \mathbb{C} \arrow[r, "\psi"] \arrow[dr, "\mathrm{proj}_1"'] & \pi^{-1}(U) \arrow[d, "\pi"] \\ & U \end{tikzcd}$$

commutes, where proj_1 is projection on the first factor, and for each $p \in U$ the map $z \mapsto \psi(p, z)$ is a $\mathbb{C}$-linear isomorphism.

The manifold $\mathbf{L}$ is called the total space, M is called the base space and each pair (U, ψ) is called a local trivialisation. A section on an open subset U is a smooth map $s \colon U \to \mathbb{L}$ such that $\pi \circ s = \mathrm{id}_U$. Suppose (U, ψ) is a local trivialisation. As for each $p \in U$ the map $z \mapsto \psi(p, z)$ is a $\mathbb{C}$-linear isomorphism, it is given by multiplication by an element $s_U(p) \in \mathbf{L}_p \backslash 0 \cong \mathbb{C}^\times$. Then the map, a local section, $s_U \colon U \to \mathbf{L}$ is smooth is nowhere vanishing on U. We call such a map a local frame. This means that for any section σ its restriction to U is given by $\sigma|_U = f_U s_U$ where $f_U \in \mathrm{C}^\infty(M) \otimes \mathbb{C} =: \mathrm{C}^\infty_{\mathbb{C}}(M)$ is a smooth complex-valued function on M. If (V, ϕ) is another local trivialisation, there exists a smooth map $g_{UV} \colon U \cap V \to \mathbb{C}^\times$ such that on $U \cap V$ we have $s_U = g_{UV} s_V$. See [12] for more details.

So we see that there is a correspondence between complex valued functions on a trivialisation U and sections of $\mathbf{L}$ on that trivialisation. Thus the Hilbert space we want will have the underlying vector space as $\mathcal{C}(M, \mathbf{L})$, the completion of[7] $\Gamma(M, \mathbf{L})$, with respect to some inner product we define soon. In order for the vector fields X_f to act on $\Gamma(M, \mathbf{L})$ we will require a notion of how to differentiate sections.

Given a complex line bundle $\pi \colon \mathbf{L} \to M$ define:

1. $\Omega^p_{\mathbb{C}}(M) = \Gamma(M, \bigwedge^p \mathrm{T}^*(M)_{\mathbb{C}})$ the smooth complex p-forms on M. If $p = 0$ we have that $\Omega^0_{\mathbb{C}}(M) = \mathrm{C}^\infty_{\mathbb{C}}(M) = \mathrm{C}^\infty(M) \otimes \mathbb{C}$.

2. $\Omega^p_{\mathbb{C}}(M, \mathbf{L}) = \Gamma(M, \bigwedge^p \mathrm{T}^*(M)_{\mathbb{C}} \otimes \mathbf{L})$ the smooth complex p-forms on M with values in $\mathbf{L}$. If $p = 0$ we have that $\Omega^0_{\mathbb{C}}(M, \mathbf{L}) = \Gamma(M, \mathbf{L})$.

N.B. that the ordinary real-linear exterior derivative d on $\Omega^\bullet(M)$ can be extended to a complex-linear exterior derivative on $\Omega^\bullet_{\mathbb{C}}(M)$, see [12].

Connections

Definition 4.2. A connection on a complex line bundle $\pi \colon \mathbf{L} \to M$ is a $\mathbb{C}$-linear map

$$\nabla \colon \Gamma(M, \mathbf{L}) \to \Omega^1_{\mathbb{C}}(M, \mathbf{L})$$

[7] generally one would require the sections to have compact support

such that for $f \in C^\infty_{\mathbb{C}}(M)$ and $s \in \Gamma(M, \mathbb{L})$ we have that it satisfies a 'product rule'

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s.$$

It is almost immediate from the product rule that ∇ is a local operator: it is enough to show that if a section s is zero on some open set U of M then $\nabla s|p$ is zero for every $p \in U$. Suppose then that $p \in U$ and σ is zero on U. Then choose some bump function ϕ which is compactly supported in U and such that $\phi(p) = 1$. Then we have that ϕs is identically zero on M. Thus as ∇ is linear $\nabla(\phi s) = 0$. Then applying the product rule we see that

$$0 = \nabla(\phi s) = \mathrm{d}\phi \otimes s + \phi \nabla s.$$

But we have that $\mathrm{d}\phi$ also has support contained in U, hence $\phi \nabla s = 0$, and evaluating at p we have $\nabla s(p) = 0$, therefore ∇ is a local operator and we will not differentiate between ∇ and $\nabla|_U$.

Let us look then at how ∇ acts locally. Suppose s_U is a frame defined on U. Then

$$\nabla(s_U) = \alpha_U \otimes s_U \text{ for some } \alpha_U \in \Omega^1_{\mathbb{C}}(U)$$

where α_U is called the connection 1-form of ∇ with respect to the local frame s_U. Now suppose σ is a section, and let its restriction to U in terms of the frame s_U be $f_U s_U$. Then we have

$$\nabla(\sigma|_U) = \nabla(f_U s_U) = (\mathrm{d}f_U + f_U \alpha_U) \otimes s_U,$$

so on U we have $\nabla = d + \alpha_U$.

There seems to be a similarity between ∇, which takes sections to 1-forms with values in $\mathbf{L}$, and the exterior derivative, which takes functions to 1-forms on M with values in $\mathbb{C}$. We want to extend how vector fields are derivations of smooth functions to 'derivations' of sections. Notice that for a vector field X and smooth function f we have that $X(f) = \mathrm{d}f(X) = \iota_X \, \mathrm{d}f$. Applying the same idea to ∇ we define

$$\nabla_X s := \iota_X \nabla s \in \Gamma(M, \mathbf{L})$$

for $X \in \mathfrak{X}_{\mathbb{C}}(M)$ and $s \in \Gamma(M, \mathbf{L})$, which we call the covariant derivative of s in the direction of X. Then linearity of the interior product gives us that $\nabla_X s$ is $C^\infty_{\mathbb{C}}(M)$-linear in X. Suppose that $X = \sum X_i \partial/\partial x_j$ in some coordinate patch U. Then

$$\nabla_X s = \nabla_{\sum X_i \partial/\partial x_j} s = \sum X_i \nabla_{\partial/\partial x_j} s$$

thus the value of $\nabla_X s$ at p only depends on s in some arbitrary neighbourhood of p and the value of X at p.

We can extend a connection ∇ uniquely to a $\mathbb{C}$-linear map

$$\nabla\colon \Omega^p_{\mathbb{C}}(M,\mathbf{L}) \to \Omega^{p+1}_{\mathbb{C}}(M,\mathbf{L})$$

by setting

$$\nabla(\beta\otimes\sigma) = \mathrm{d}\beta\otimes\sigma + (-1)^p\beta\wedge\nabla\sigma \text{ for } \beta\in\Omega^p_{\mathbb{C}}(M) \text{ and } \sigma\in\Gamma(M,\mathbf{L}).$$

Consider again the local frame s_U from before:

$$\begin{aligned}\nabla^2(s_U) &= \nabla(\alpha_U\otimes s_U)\\ &= \mathrm{d}\alpha_U\otimes s_U - \alpha_U\wedge\nabla(s_U)\\ &= \mathrm{d}\alpha_U\otimes s_U - \alpha_U\wedge\alpha_U\otimes s_U\\ &= \mathrm{d}\alpha_U\otimes s_U\end{aligned}$$

so unless all the connection 1-forms are exact, ∇^2 is non-zero — in contrast to the exterior derivative. Now suppose s_V is another frame on V so that there is a transition function g_{UV} such that $s_U = g_{UV}s_V$. Then on $U\cap V$ we have

$$\begin{aligned}\nabla^2 s_U &= \nabla^2(g_{UV}s_V)\\ &= \nabla(\mathrm{d}g_{UV}\otimes s_V + g_{UV}\wedge\alpha_V\otimes s_V)\\ &= \mathrm{dd}g_{UV}\otimes s_V - \mathrm{d}g_{UV}\wedge\alpha_V\otimes s_V\\ &\qquad + \mathrm{d}g_{UV}\wedge\alpha_V\otimes s_V - g_{UV}\alpha_V\wedge\alpha_V\otimes s_V\\ &= -\mathrm{d}g_{UV}\wedge\alpha_V\otimes s_V + \mathrm{d}g_{UV}\wedge\alpha_V\otimes s_V + g_{UV}\,\mathrm{d}\alpha_V\otimes s_V\\ &= g_{UV}\,\mathrm{d}\alpha_V\otimes s_V.\end{aligned}$$

Hence

$$\mathrm{d}\alpha_U\otimes g_{UV}\otimes s_V = g_{UV}\,\mathrm{d}\alpha_V\otimes s_V$$

thus as s_V and g_{UV} are non-vanishing on $U\cap V$, we have $\mathrm{d}\alpha_U = \mathrm{d}\alpha_V$ on $U\cap V$. Accordingly, there exists a globally defined 2-form Ω such that for any frame s_U we have $\nabla^2(s_U) = \Omega\otimes s_U$. N.B. that although on each trivialising neighbourhood U we have $\Omega = \mathrm{d}\alpha_U$, Ω is not exact as (for non-trivial line bundles) α_U is not globally defined. However it is then locally exact thus is closed, a local property.

Ω is called the curvature form of the connection. As mentioned before, we are more interested in using the connection to apply vector fields to sections. So how does this curvature form act on vector fields? Let s_U be a locally defined frame and $X, Y \in \mathfrak{X}_{\mathbb{C}}(M)$. Then

$$\begin{aligned}
\Omega(X,Y) \otimes s_U &= \iota_Y \iota_X \,\mathrm{d}\alpha_U \otimes s_U \\
&= \iota_X \,\mathrm{d}\iota_Y \alpha_U \otimes s_U - \iota_Y \,\mathrm{d}\iota_X \alpha_U \otimes s_U \\
&\quad - \iota_{[X,Y]} \alpha_U \otimes s_U \\
&= \nabla_X \nabla_Y (s_u) - \alpha_U(Y)\alpha_U(X) s_U \\
&\quad - \nabla_Y \nabla_X (s_U) + \alpha_U(X)\alpha_U(Y) s_U - \nabla_{[X,Y]} s_U.
\end{aligned}$$

Thus globally we have $\iota_Y \iota_X \nabla^2 = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$. The percipient reader may already see the similarity[8] between this and how we tried to fix $\mathcal{G}$ earlier.

Hermitian Structure

We are interested in complex line bundles with more geometric structure.

A hermitian structure on a complex line bundle $\mathbf{L} \xrightarrow{\pi} M$ is a smooth choice of hermitian metrics on each fibre of the bundle. Explicitly, for each $x \in M$, $h_x(\cdot,\cdot)$ is a hermitian metric on $\mathbf{L}_x$: For each $u, v \in \mathbf{L}_x$

- $h_x(u,v)$ is linear in u,
- $h_x(u,v) = \overline{h_x(v,u)}$,
- $h_x(u,u) > 0$ for $u \neq 0$,

and if σ, η are sections, then $h(\sigma,\eta)$ is a smooth function.

Then a connection ∇ is said to be a hermitian connection compatible with h (or a h-connection) if for all sections σ, η we have

$$\mathrm{d}h(\sigma,\eta) = h(\nabla\sigma,\eta) + h(\sigma,\nabla\eta).$$

For each frame s_U define the (real) function $h_U := h(s_U, s_U)$. We see that

$$\begin{aligned}
\mathrm{d}h_U &= h(\nabla s_U, s_U) + h(s_U, \nabla s_U) \\
&= h(\alpha_U \otimes s_U, s_U) + \overline{h(\alpha_U \otimes s_U, s_U)} \\
&= h_U(\alpha_U + \overline{\alpha_U}).
\end{aligned}$$

Differentiating both sides we have

$$\begin{aligned}
0 &= \mathrm{d}h_U \wedge (\alpha_U + \overline{\alpha_U}) + h_U(\mathrm{d}\alpha_U + \mathrm{d}\overline{\alpha_U}) \\
&= h_U(\alpha_U + \overline{\alpha_U}) \wedge (\alpha_U + \overline{\alpha_U}) + h_U(\mathrm{d}\alpha_U + \mathrm{d}\overline{\alpha_U}) \\
&= h_U(\mathrm{d}\alpha_U + \mathrm{d}\overline{\alpha_U}).
\end{aligned}$$

[8] Or rather, connection.

However as d is actually a complexified real-linear function we have $\mathrm{d}\overline{\alpha_U} = \overline{\mathrm{d}\alpha_U}$ thus

$$\mathfrak{Im}(\mathrm{d}\alpha_U) = \mathrm{d}\alpha_U + \overline{\mathrm{d}\alpha_U} = 0$$

hence $\mathrm{d}\alpha_U$ is purely imaginary-valued, and thus so is Ω. Indeed, if s_U is a unitary frame, i.e. $h_U = \mathbf{1}$, then $\mathrm{d}h_U = 0$ hence α_U is also purely imaginary-valued. We can always normalise a frame to make it unitary.

If the base manifold is complex, then we can also consider holomorphic line bundles where the total space and projection maps are holomorphic.

So far we have not actually shown a connection actually exists, but we will prove this for Delzant spaces later.

4.3 Prequantisation

Theorem 4.3 (adapted from [4]). *Let $\mathbf{L}$ be a complex line bundle over a compact $2n$-dimensional symplectic manifold (M, ω) with hermitian metric h and a h-connection ∇ with curvature form $\Omega = (i/\hbar)\omega$. Then let $L^2(M, \mathbf{L})$ be the completion of $\Gamma(M, \mathbf{L})$ with respect to the inner product*

$$\langle s_1, s_2 \rangle = \int_M h(s_1, s_2) \frac{\omega^n}{n!}.$$

Then $L^2(M, \mathbf{L})$ is a Hilbert space and we have a mapping

$$\mathcal{G} \colon f \mapsto i\hbar \nabla_{X_f} + f$$

that sends smooth functions on M to self-adjoint operators on $L^2(M, \mathbf{L})$ such that for all $f, g \in \mathrm{C}^\infty(M)$ and $\lambda, \mu \in \mathbb{R}$ we have

1. $\mathcal{Q}(\lambda f + \mu g) = \lambda \mathcal{Q}(f) + \mu \mathcal{Q}(g)$
2. $\mathcal{Q}(\mathbf{1}) = \mathrm{id}_{L^2(M,\mathbf{L})}$
3. $[\mathcal{Q}(f), \mathcal{Q}(g)] = i\hbar \mathcal{Q}(\{f, g\})$.

Then we say we have pre-quantised (M, ω).

Proof. **Q**1 follows from the linearity of assigning Hamiltonian vector fields and of ∇. **Q**2 is clear. Let us now show we have finally solved **Q**3. Let

s, s_1, s_2 be sections. We have

$$
\begin{aligned}
[\mathcal{Q}(f), \mathcal{Q}(g)](s) &= [i\hbar\nabla_{X_f} + f, i\hbar\nabla_{X_g} + g](s) \\
&= -\hbar^2[\nabla_{X_f}, \nabla_{X_g}](s) + i\hbar[\nabla_{X_f}, g](s) + i\hbar[f, \nabla_{X_g}](s) \\
&= -\hbar^2(\nabla_{X_f}\nabla_{X_g} - \nabla_{X_g}\nabla_{X_f})(s) \\
&\qquad + i\hbar(\nabla_{X_f}(gs) - g\nabla X_f s - \nabla_{X_g}(fs) + f\nabla_{X_g} s) && \text{[a]} \\
&= -\hbar^2(\nabla_{[X_f, X_g]} s + \Omega(X_f, X_g)s) \\
&\qquad + i\hbar(X_f(g)s + g\nabla_{X_f} s && \text{[b]} \\
&\qquad\qquad - g\nabla X_f s - X_g(f)s - f\nabla_{X_g} s + f\nabla_{X_g} s) \\
&= -\hbar^2\nabla_{X_{\{f,g\}}} s - i\hbar\omega(X_f, X_g)s + 2i\hbar\{f, g\}s \\
&= i\hbar(i\hbar\nabla_{X_{\{f,g\}}} s + \{f, g\}s) \\
&= i\hbar\mathcal{Q}(\{f, g\})(s).
\end{aligned}
$$

Line [b] follows from line [a] as $\iota_Y\iota_X\nabla^2 = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$.

Now to show $\mathcal{Q}(f)$ is self-adjoint. We make two preliminary observations. Firstly, f is real thus $h(fs_1, s_2) = h(s_1, fs_2)$. Secondly, ∇ is a h-connection thus

$$X_f(h(s_1, s_2)) = \iota_{X_f}\,\mathrm{d}h(s_1, s_2) = h(\nabla_{X_f} s_1, s_2) + h(s_1, \nabla_{X_f} s_2).$$

We have

$$
\begin{aligned}
\langle\mathcal{Q}(f)s_1, s_2\rangle &= \int_M h(i\hbar\nabla_{X_f} s_1 + fs_1, s_2)\frac{\omega^n}{n!} \\
&= i\hbar\int_M X_f(h(s_1, s_2))\frac{\omega^n}{n!} \\
&\qquad - i\hbar\int_M h(s_1, \nabla_{X_f} s_2)\frac{\omega^n}{n!} \\
&\qquad\qquad + \int_M h(s_1, fs_2)\frac{\omega^n}{n!}.
\end{aligned}
$$

Therefore to show $\mathcal{Q}(f)$ is self-adjoint all we need to show is that

$$\int_M X_f(g)\frac{\omega^n}{n!} = 0$$

where $g = h(s_1, s_2)$. We know that X_f is Hamiltonian thus $\mathcal{L}_{X_f}\omega = 0$. Ergo

$$\mathcal{L}_{X_f}\omega^n = (\mathcal{L}_{X_f}\omega^{n-1}) \wedge \omega + \omega^{n-1} \wedge \mathcal{L}_{X_f}\omega = (\mathcal{L}_{X_f}\omega^{n-1}) \wedge \omega$$

so by induction we have that $\mathcal{L}_{X_f}\omega^n = 0$. Hence

$$\mathcal{L}_{X_f}\left(g\frac{\omega^n}{n!}\right) = (\mathcal{L}_{X_f}g)\frac{\omega^n}{n!} + g\mathcal{L}_{X_f}\left(\frac{\omega}{n!}\right) = (\mathcal{L}_{X_f}g)\frac{\omega^n}{n!} = X_f(g)\frac{\omega^n}{n!}.$$

But then $g\omega^n/n!$ is a top-form thus

$$X_f(g)\frac{\omega^n}{n!} = \mathcal{L}_{X_f}\left(g\frac{\omega^n}{n!}\right) = \iota_{X_f}\,\mathrm{d}\left(g\frac{\omega^n}{n!}\right) + \mathrm{d}\left(\iota_{X_f}g\frac{\omega^n}{n!}\right) = \mathrm{d}\left(\iota_{X_f}g\frac{\omega^n}{n!}\right).$$

Therefore, as M is compact and without boundary, Stoke's theorem gives us

$$\int_M X_f(g)\frac{\omega^n}{n!} = \int_M \mathrm{d}\left(\iota_{X_f}g\frac{\omega^n}{n!}\right) = 0$$

$\square$

4.4 Line Bundles over Delzant Spaces

(adapted from [1] chapter VII). Delzant spaces can be described combinatorially and as we shall now show, we can describe a nice class of holomorphic line bundles over them in a similar manner.

Let Δ be a Delzant polytope. Recall that in the complex construction we associated to each face $\mathcal{F}$ a subset $I_{\mathcal{F}} \subseteq \{1, \ldots, d\}$. These sets together with the map $\mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^n$ is all we needed for the complex construction of M_Δ.

Choose some linear map $g : \mathbb{R}^d \to \mathbb{R}$ that is integral, in the sense that $g(\mathbb{Z}^n) \subseteq \mathbb{Z}$. Set

$$\begin{aligned} u_j^g &= (u_j, g(e_j)) \text{ for } j = 1, \ldots, d \\ u_{d+1}^g &= (0, 1) \end{aligned}$$

and for each face $\mathcal{F}$

$$\begin{aligned} I_{\mathcal{F}}^g &= I_{\mathcal{F}} \\ I_{\mathcal{F}+}^g &= I_{\mathcal{F}} \cup \{d+1\} \end{aligned}$$

and for the maps ι, π set

$$\pi^g(e_j) = u_j^g, \qquad \iota^g = (\iota, -g \circ \iota).$$

And so we have a short exact sequence

$$0 \longrightarrow \mathbb{R}^m \xrightarrow{\iota^g} \mathbb{R}^{d+1} \xrightarrow{\pi^g} \mathbb{R}^{n+1} \longrightarrow 0\ .$$

Then as before, define

$$\mathbb{C}^{d+1}|_{\mathring{\mathcal{F}}} = \{w \in \mathbb{C}^{d+1} : w_j = 0 \iff j \in I_{\mathcal{F}}\},$$

and

$$\mathbb{C}^{d+1}|_{\mathring{\mathcal{F}}+} = \{w \in \mathbb{C}^{d+1} : w_j = 0 \iff j \in I_{\mathcal{F}+}\},$$

and then

$$\mathbb{C}^{d+1}|^{g}_{\Delta} = \bigcup_{\mathcal{F} \text{ face of } \Delta} (\mathbb{C}^{d+1}|_{\mathring{\mathcal{F}}} \cup \mathbb{C}^{d+1}|_{\mathring{\mathcal{F}}+}).$$

Thus to actually complete the construction in the same way (although our construction will not be compact of course) we only need to show that $\mathbb{T}^m_{\mathbb{C}}$ acts freely on $\mathbb{C}^{d+1}|^{g}_{\Delta}$. Now a 'vertex' in this set up is given by $I_{\mathcal{F}+}$ where $\mathcal{F}$ is a vertex of Δ. Then suppose

$$I_{\mathcal{F}+} = \{j_1, \ldots, j_n, d+1\}.$$

Now, if we show for each vertex $\mathcal{F}$ the vectors $u^g_{j_1}, \ldots, u^g_{j_n}, u^g_{d+1}$ are a $\mathbb{Z}$-basis for $\mathbb{Z}^{n+1}$, then we can use the same argument as in the complex construction to show that $\mathbb{T}^m_{\mathbb{C}}$ acts freely on $\mathbb{C}^{d+1}|^{g}_{\Delta}$. However we know that $u_{j_1}, \ldots, u_{j_d}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$. Thus

$$(u_{j_1}, 0), \ldots, (u_{j_d}, 0), (0, 1)$$

is a $\mathbb{Z}$-basis for $\mathbb{Z}^{n+1}$. But each $g(e_j)$ is an integer so the $\mathbb{Z}$-span of the vectors above is the same as

$$(u_{j_1}, g(e_{j_1})), \ldots, (u_{j_d}, g(e_{j_n})), (0, 1)$$

so we are done. Then denote $\mathbb{C}^{d+1}|^{g}_{\Delta}/\mathbb{T}^m_{\mathbb{C}}$ by M^g_{Δ}. Thus M^g_{Δ} is a complex manifold with an effective $\mathbb{T}^{n+1}$ action.

Now we need to show that M^g_{Δ} is actually a holomorphic line bundle over M_{Δ}.

First notice that

$$\mathbb{C}^{d+1}|_{\mathring{\mathcal{F}}} = \mathbb{C}^{d}|_{\mathring{\mathcal{F}}} \times \mathbb{C}^{\times}$$

and

$$\mathbb{C}^{d+1}|_{\mathring{\mathcal{F}}+} = \mathbb{C}^{d}|_{\mathring{\mathcal{F}}} \times 0.$$

Thus $\mathbb{C}^{d+1}|^{g}_{\Delta} = \mathbb{C}^{d}|_{\Delta} \times \mathbb{C}$, so we have a projection map $q\colon \mathbb{C}^{d+1}|^{g}_{\Delta} \to \mathbb{C}^{d}|_{\Delta}$ with fibres $\mathbb{C}$. Then the $\mathbb{T}^m_{\mathbb{C}}$ action[9] on $\mathbb{C}^{d+1}|^{g}_{\Delta}$ is

$$v \cdot (w_1, \ldots, w_d, w_{d+1}) = (e^{2\pi i\iota(v_1)}w_1, \ldots, e^{2\pi i\iota(v_d)}w_d, \frac{w_{d+1}}{e^{2\pi i g(\iota(v))}})$$

[9]For notational ease we have replaced $\tilde{\iota_{\mathbb{C}}}$ with ι.

thus q is $\mathbb{T}^m_{\mathbb{C}}$-equivariant and so descends to the quotient

$$\begin{array}{ccc} \mathbb{C}^{d+1}|^g_\Delta & \xrightarrow{q} & \mathbb{C}^d|_\Delta \\ \downarrow & & \downarrow \\ M^g_\Delta & \longrightarrow & M_\Delta \end{array}$$

and so we can see that M^g_Δ is indeed a holomorphic complex line bundle over M_Δ.

Example 4.4. Here we consider $\mathbb{C}\mathrm{P}^n$ as constructed in e3.8. Let $m \in \mathbb{Z}$ and define

$$\begin{aligned} g\colon \mathbb{R}^{n+1} &\to \mathbb{R} \\ e_j &\mapsto 0 \qquad \text{for } 0 \leq j \leq n \\ e_{n+1} &\mapsto m \qquad\qquad\qquad . \end{aligned}$$

Then $\mathbb{C}^\times$ acts on $\mathbb{C}^{n+2}|^g_{\Delta^n}$ by

$$\alpha \cdot (w_1, \ldots, w_{n+1}, w_{n+2}) = (\alpha w_1, \ldots, \alpha w_{n+1}, \alpha^{-m} w_{n+2})$$

Thus $M^g_{\Delta^n}$ is the $\mathbb{C}\mathrm{P}^n$ version of line bundle $\mathbb{L}_{(-m)}$ over $\mathbb{C}\mathrm{P}^n$.

Example 4.5. Let Δ^n_{H} be as in the Hirzebruch example 3.9. Then define

$$\begin{aligned} g\colon \mathbb{R}^4 &\to \mathbb{R} \\ e_1, e_3 &\mapsto 0 \\ e_2 &\mapsto -n-1 \\ e_4 &\mapsto -1. \end{aligned}$$

Then $(\mathbb{C}^\times)^2$ acts on $\mathbb{C}^5|^g_{\Delta^n_{\mathrm{H}}}$ by

$$(\alpha_1 \alpha_2) \cdot (z_1, z_2, z_3, z_4, z_5) = (\alpha_1 z_1, \alpha_1 z_2, \alpha_2 z_3, \alpha_2 \alpha_2^{-n} z_4, \alpha_1 \alpha_2 z_5).$$

Principal Circle Bundles

Now we want to construct a line bundle as in the last section and build a connection on it where the curvature is related to ω_Δ. However to do this we will have to use the concept of a principal $\mathbb{S}^1$-bundle. In fact there is a $1-1$ correspondence between $\mathbb{S}^1$-bundles and hermitian line bundles (one can see why this might be true by considering the unit vectors in each fibre). We do not have the space to fully develop the relationship between principal bundles and hermitian line bundles so the following is more of a sketch proof. See [10] and [7].

Theorem 4.6. *For Δ a Delzant polytope with $\lambda = (\lambda_1, \ldots, \lambda_d)$ integral, then M_Δ has a holomorphic line bundle with a hermitian connection with curvature $2\pi i \omega_\Delta$.*

Proof. Let us add some new notation to the (symplectic) Delzant construction. Define $\lambda = (\lambda_1, \ldots, \lambda_d) \in (\mathbb{R}^d)^*$ and $\eta = \iota^*(\lambda)$, which is integral as λ is. Then define $\nu = \mu - \eta$. Then $\mathrm{d}\eta = \mathrm{d}\mu$ and $\nu^{-1}(-\eta) = \mu^{-1}(0)$. Here then we are just shifting μ and then reducing at where 0 is sent to, so the Delzant space is the same. Then

$$\begin{aligned} \phi \colon \mathbb{C}^d &\to (\mathbb{R}^d)^* \\ (z_1, \ldots, z_d) &\mapsto \frac{1}{2}(|z_1|^2, \ldots, |z_d|^2) \end{aligned}$$

is the moment map for the $\mathbb{T}^d$ action on $\mathbb{C}^d$ and we have that $\nu = \iota^* \circ \phi$. Let

$$\omega_0 = \sum \mathrm{d}x_j \wedge \mathrm{d}y_j$$

be the standard symplectic form on $\mathbb{C}^d$. Then recall that for $X^j = e_j \in \mathbb{R}^d$ its fundamental vector is

$$X^j_\phi = y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j}.$$

Then there exists a potential form for ω_0

$$\alpha_0 = \frac{1}{2} \sum (x_j \, \mathrm{d}y_j - y_j \, \mathrm{d}x_j)$$

such that

$$X^j_\phi \lrcorner \, \alpha_0 = \frac{1}{2}(x_j^2 + y_j^2) = \tilde{\phi}_{X_j}$$

for each $j = 1, \ldots, d$. Then we also have that[10] $\tau^* \alpha_0$ is a potential form for $\tau^* \omega_0$ and if $Y \in \mathbb{R}^m$ then

$$Y_\nu \lrcorner \, \tau^* \alpha_0 = \tilde{\nu}_Y.$$

Define $P = \nu^{-1}(-\eta) \times \mathbb{S}^1$. Then P is a trivial $\mathbb{S}^1$ bundle over $\nu^{-1}(-\eta)$. Then define the 1-form[11] on P

$$\Theta = 2\pi i \tau^* \alpha_0 + \mathrm{d}\theta$$

and we have

$$\Theta(\frac{\partial}{\partial \theta}) = 1$$

[10] τ is the inclusion of $\nu^{-1}(\eta)$

[11] strictly speaking we are taking the respective pullbacks of $\tau^* \alpha_0$ and $\mathrm{d}\theta$ to P

thus Θ is a connection form with respect to $\mathbb{S}^1$. It's curvature form is

$$\mathrm{d}\Theta = 2\pi i \tau^* \omega_0.$$

P also has a free $\mathbb{T}^m$ action where $a \in \mathbb{T}^m$ acts by $a \cdot (z, w) = (a \cdot z, a \cdot w)$ where $a \cdot w$ is the $\mathbb{T}^m$ action on $\mathbb{S}^1$ with weight $-\eta$ i.e.

$$a \cdot w = e^{-2\pi\eta(a)} w.$$

N.B. that this action only makes sense as η is integral. Thus $P \to P/\mathbb{T}^m$ is a principal $\mathbb{T}^m$ bundle. We have that Θ is preserved by the $\mathbb{T}^m$ action as

$$\mathcal{L}_{Y_\nu} \tau^* \alpha_0 = Y_\nu \lrcorner \mathrm{d}\tau^* \alpha_0 + \mathrm{d} Y_\nu \lrcorner \tau^* \alpha_0 = -\mathrm{d}\tilde{\nu_Y} + \mathrm{d}\tilde{\nu_Y} = 0.$$

For the action on $\mathbb{S}^1$, the fundamental vector field for $Y \in \mathbb{R}^m$ is

$$-2\pi i \eta(Y) \frac{\partial}{\partial \theta}.$$

Then for $Y \in \mathbb{R}^m$, the fundamental vector field Y_P on P is

$$Y_P = Y_\nu - 2\pi i \eta(Y) \frac{\partial}{\partial \theta}.$$

But then

$$\Theta(Y_P) = 2\pi i \tilde{\nu}_Y - 2\pi\eta(Y)$$

and for $z \in \nu^{-1}(-\eta)$ we have $\tilde{\nu}_Y(z) = \eta(Y)$ thus $\Theta(Y_P) = 0$. Hence Θ is a horizontal form on P as a $\mathbb{T}^m$ bundle, and so descends to a connection 1-form Ξ on the principal $\mathbb{S}^1$-bundle $(\nu^{-1}(-\eta) \times \mathbb{S}^1)/\mathbb{T}^m \to \nu^{-1}(-\eta)/\mathbb{T}^m$. However the diagram

$$\begin{array}{ccc} \nu^{-1}(-\eta) \times \mathbb{S}^1 & \longrightarrow & \nu^{-1}(-\eta) \\ \downarrow & & \downarrow \\ (\nu^{-1}(-\eta) \times \mathbb{S}^1)/\mathbb{T}^m & \longrightarrow & \nu^{-1}(-\eta)/\mathbb{T}^m \end{array}$$

commutes thus the curvature form $\mathrm{d}\Xi$ is precisely $2\pi i \omega_\Delta$.

Thus $\mathcal{Z} \times_{\mathbb{T}^m} \mathbb{S}^1 \to M_\Delta$ is a principal $\mathbb{S}^1$ bundle with connection Ξ and curvature $2\pi i \omega_\Delta$. From this principal $\mathbb{S}^1$-bundle one can construct an associated hermitian line bundle

$$(\mathcal{Z} \times_{\mathbb{T}^m} \mathbb{S}^1) \times_{\mathbb{S}^1} \mathbb{C}.$$

We can view $\mathcal{Z} \times_{\mathbb{T}^m} \mathbb{S}^1$ as the principal $\mathbb{C}^\times$-bundle

$$\mathbb{C}^d|_\Delta \times_{\mathbb{T}^m_{\mathbb{C}}} \mathbb{C}^\times$$

and so we can see that the associated hermitian line bundle is

$$(\mathbb{C}^d|_\Delta \times_{\mathbb{T}^m_{\mathbb{C}}} \mathbb{C}^\times) \times_{\mathbb{C}^\times} \mathbb{C} = \mathbb{C}^d|_\Delta \times_{\mathbb{T}^m_{\mathbb{C}}} \mathbb{C} = M^g_\Delta$$

where g is the map defined by $g\colon e_j \mapsto \lambda_j$. Then (see [10] section III.3) the connection Ξ on $\mathcal{Z} \times_{\mathbb{T}^m} \mathbb{S}^1$ determines a connection on the hermitian line bundle $(\mathcal{Z} \times_{\mathbb{T}^m} \mathbb{S}^1) \times_{\mathbb{S}^1} \mathbb{C}$ with curvature form $2\pi i \omega_\Delta$. □

Thus to use theorem 4.3 we need to scale the symplectic form:

Corollary 4.7. *Given a Delzant space* M_Δ, ω_Δ *where* λ *is integral, we can pre-quantise* $(M_\Delta, 2\pi\hbar\omega_\Delta)$ *with the line bundle defined by* $g\colon e_j \mapsto \lambda_j$.

4.5 Kähler Polarisation

So far we have managed to build a prequantisation of our Delzant spaces. The reader may be wondering why this is a *pre*quantisation — what then is a quantisation?

Definition 4.8. Let M, ω be a symplectic manifold of dimension $2n$. A completely integrable system is a collection of n functions $f_1, \dots, f_n$ such that they pairwise Poisson commute and their differentials are linearly independent on an open dense subset of M.

Example 4.9. Suppose M_Δ is a Delzant space of dimension $2n$. Then it has an effective Hamiltonian $\mathbb{T}^n$ action with moment map

$$\mu_\Delta = (f_1, \dots, f_n)\colon M_\Delta \to (\mathbb{R}^n)^* \cong \mathbb{R}^n.$$

Then straight away we have that $f_1, \dots, f_n$ Poisson commute. Let $p \in M$ and suppose $X_1, \dots, X_n$ are the standard basis vectors of $\mathbb{R}^n$ so that

$$\mathrm{d}f_j|_p = \omega_p(v, \underline{X_j}|_p) \text{ for all } v \in \mathrm{T}_p\, M.$$

Thus if we had some scalars $a_1, \dots a_n$ such that $\sum_{j=1}^n a_j\, \mathrm{d}f_j|_p(v) = 0$ for all $v \in \mathrm{T}_p\, M$ then

$$\omega_p(v, \sum_{j=1}^n a_j \underline{X_j}|_p) = 0 \text{ for all } v \in \mathrm{T}_p\, M$$

and so by non-degeneracy of ω we have that $\sum_{j=1}^n a_j \underline{X_j}|_p = 0$. However, one can see from the Delzant construction that there is an open dense subset of $\mathbb{C}^d$ where $\mathbb{T}^d$ acts freely, namely the preimage of the interior of Δ. Under the quotient, $\mathbb{T}^n$ still acts freely on this open dense subset of M_Δ. Thus if p is in this set we have $\sum_{j=1}^n a_j X_j = 0$ thus the $a_1, \dots, a_n$ are zero and so we have that $f_1, \dots, f_n$ form an integrable system.

The problem with just taking a prequantisation is that the space of sections $\mathrm{L}^2(M, \mathbf{L})$ is to 'big'. If we have a completely integrable system as above, the operators that they get sent to do not form a complete set of quantum operators (see [4] for the quantum definition). This means that we may have a classical system that we can fully solve but not its corresponding quantum system. The work around is to attempt to half the Hilbert space using something called a polarisation. In general this is a complicated process which makes the quantum system even more esoteric however for Kähler manifolds all we have to do is restrict ourselves to holomorphic sections $\Gamma_{\mathrm{hol}}(M, \mathbf{L})$ of a prequantum holomorphic line bundle (see [4]). We shall now see that in the case of Delzant spaces there is a straightforward way to calculate the dimension of this new Hilbert space.

Lemma 4.10. *For a Delzant polytope Δ we have that $\mathbb{C}^d \setminus \mathbb{C}^d|_\Delta$ is a union of complex submanifolds of $\mathbb{C}^d$ of codimension at least 2.*

Proof. If $z \in \mathbb{C}^d \setminus \mathbb{C}^d|_\Delta$ then it cannot have all non-zero coordinates as otherwise it would be in $\mathbb{C}^d|_{\mathring{\Delta}}$. Furthermore the indices of its zero coordinates cannot be $I_{\mathcal{F}}$ for any face $\mathcal{F}$ of Δ. In particular each $j = 1, \ldots, d$, the face defined by u_j is in Δ thus z must have at least two zero coordinates thus is an element of a complex codimension 2 submanifold of $\mathbb{C}^d$. $\square$

Lemma 4.11. *If Δ is a Delzant polytope, then any holomorphic function*

$$s\colon \mathbb{C}^d|_\Delta \to \mathbb{C}$$

can be extended uniquely to a holomorphic function on all of $\mathbb{C}^d$.

Proof. This follows from lemma 4.10 and the Second Riemann Theorem as seen in [5] which states that for an analytic set A in $D \subseteq \mathbb{C}^d$ such that $\dim A \leq d-2$ then every holomorphic function f in $D \setminus A$ has a unique holomorphic extension $\tilde{f}$ to D. $\square$

Theorem 4.12 ([8])**.** *Let Δ be a Delzant polytope with λ integral, and M^g_Δ the holomorphic line bundle over M_Δ as in REF. Then the dimension of $\Gamma_{hol}(M_\Delta, M^g_\Delta)$ is equal to $\mathbb{Z}^d_{\geq 0} \cap (\iota^*)^{-1}(\iota^*(\lambda))$.*

Proof. If s is a holomorphic section of M^g_Δ, then we can consider it as a holomorphic $\mathbb{T}^m_{\mathbb{C}}$-equivariant function

$$s\colon \mathbb{C}^d|_\Delta \to \mathbb{C}$$

and thus by 4.11 it has a unique extension

$$\tilde{s}\colon \mathbb{C}^d \to \mathbb{C}$$

where it is $\mathbb{T}_{\mathbb{C}}^m$-equivariant on $\mathbb{C}^d|_\Delta$ which is dense in $\mathbb{T}^d$ thus is $\mathbb{T}_{\mathbb{C}}^m$-equivariant everywhere. Then write $\tilde{s}$ as its Taylor series

$$\tilde{s}(z) = \sum_{I \in \mathbb{Z}_{\geq 0}^d} a_I z^I$$

where we are using multi-index notation. Then consider just one term z^I, where $I = \{j_1, \ldots, j_d\}$. If $v \in \mathbb{T}_{\mathbb{C}}^d$ then

$$(v \cdot z)^I = (e^{2\pi i v_1} z_1, \ldots, e^{2\pi i v_d} z_d)^I = (e^{2\pi i j_1 v_1} z_1^{j_1}) \cdots (e^{2\pi i j_d v_d} z_d^{j_d}) = e^{2\pi i I(v)} z^I$$

where we view I as an (integral) element of $(\mathbb{R}^d)^*$. Then if $v \in \mathbb{T}_{\mathbb{C}}^m$ we actually have

$$(v \cdot z)^I = e^{2\pi i \iota^*(I)(v)} z^I$$

and since the action of $\mathbb{T}_{\mathbb{C}}^m$ on $\mathbb{C}$ is given by

$$v \cdot z^I = e^{-2\pi \iota^*(\lambda)(v)} z^I$$

we must have $\iota^*(I) = \iota^*(-\lambda)$ and the result follows. $\square$

Corollary 4.13. *The dimension of* $\Gamma_{hol}(M_\Delta, M_\Delta^g)$ *is equal to the number of integer lattice points in* Δ*.*

Proof. See [ref] claim 1. $\square$

Example 4.14. Finally, let us return to our two favourite examples of Delzant spaces. For example 3.8 the integer lattice points in Δ^n are the corners thus the dimension of the quantised space is $n + 1$. For the Hirzebruch surfaces, in our construction 3.9 for any value of n the vertices are the only integral points and therefore the quantised space has dimension 5.

Notation

$\mathfrak{Im}$	imaginary part
$\mathrm{T}(M), \mathrm{T}_{\mathbb{C}}(M)$	tangent space and complexified tangent space
$\mathrm{C}^\infty(M), \mathrm{C}^\infty_{\mathbb{C}}(M)$	smooth real/complex valued functions on M
$\mathfrak{X}(M), \mathfrak{X}(M)_{\mathbb{C}}$	vector fields and complexified vector fields on M.

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