Discrete Morse Theory for Persistent Cosheaf Homology

Final Honour School of Mathematics Part C
C3.9 Computation Algebraic Topology

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Given a filtration on a finite simplical complex, one can study the topological nature of how the complex changes with respect to the filtration using the tools of persistent homology[1]. On the other hand, one can attach additional data to the complex using (co)sheaves and study the topological properties of this data using (co)sheaf homology[2]. Both these techniques have seen great success in tackling applied problems, however it is of critical importance that these computations can be done efficiently, in both time and data space. To that end, discrete Morse theory has shown itself to be a valuable tool in both persistent homology for filtrations and in (co)sheaf homology. In this paper we will first give a short introduction to the theory of cosheaves on finite simplicial complexes. Subsequently, we look at how to unify persistent and cosheaf homology to the persistent homology of sequences of cosheaves on a finite simplicial complex. Finally, we will extend the discrete Morse theory to simplify such computations.

[1] [1]
[2] [2]

1 Cosheaves over Finite Simplicial Complexes

1.1 Task I

Throughout this paper X is a finite simplicial complex. Viewing X as a poset, let X^{op} be X with the opposite order. Let $\mathbb{F}$ be a field.

Definition 1.1 (Cosheaf)**.** A *cosheaf* $\mathcal{F}$ *over* X is a (covariant) functor from X^{op} to $\mathbf{FinVect}_{\mathbb{F}}$. Explicitly, for every simplex $\tau \in X$ the cosheaf $\mathcal{F}$ assigns τ a finite dimensional vector space $\mathcal{F}(\tau)$, called the *stalk* at τ, and for every face $\sigma \leq \tau$ a linear map

$$\mathcal{F}(\tau \geq \sigma) : \mathcal{F}(\tau) \to \mathcal{F}(\sigma)$$

called a *corestriction* such that for any coface $\gamma \geq \tau$ of τ we have

$$\mathcal{F}(\tau \geq \sigma) \circ \mathcal{F}(\gamma \geq \tau) = \mathcal{F}(\gamma \geq \sigma) \ .$$

We also require that $\mathcal{F}(\tau \leq \tau) = \mathrm{id}_{\mathcal{F}(\tau)}$ for every simplex τ. Denote the category[3] of cosheaves over X by $\mathbf{CoShv}_X$.

Definition 1.2 (Orientation)**.** Choose an ordering of the vertices of X so we can write the vertices as $\{v_0, \ldots, v_n\}$. Then this ordering of vertices induces an (local) orientation on all simplices of X. Then let $\tau = (v_0, \ldots, v_k)$ be a (oriented) simplex of X. Then suppose $\sigma = (v_0, \ldots, \hat{v_i}, \ldots, v_k)$ is an (oriented) codimension one face of τ, where $\hat{v_i}$ denotes that vertex has been deleted for some $i \in \{0, \ldots, k\}$. Then we define their *boundary coefficient* as $[\tau : \sigma] := (-1)^i \in \mathbb{F}$.

Definition 1.3. Let $\mathcal{F}$ be a cosheaf over X. For each $i \geq 0$ define the set of *i-chains of X with coefficients in $\mathcal{F}$* as

$$\mathrm{C}_i(X; \mathcal{F}) = \bigoplus_{\substack{\sigma \in X \\ \dim \sigma = i}} \mathcal{F}(\sigma) \ .$$

Definition 1.4. We define the *boundary operator*

$$\partial_i^{\mathcal{F}} : \mathrm{C}_i(X; \mathcal{F}) \to \mathrm{C}_{i-1}(X; \mathcal{F})$$

as

$$\partial_i^{\mathcal{F}}(v_\tau) = \sum_{\substack{\sigma \leq \tau \\ \dim \tau - \dim \sigma = 1}} [\tau : \sigma] \mathcal{F}(\tau \geq \sigma)(v_\tau)$$

for $v_\tau \in \mathcal{F}(\tau)$ and extend it linearly.

Proposition 1.5. *$(C_\bullet(X; \mathcal{F}), \partial_\bullet^{\mathcal{F}})$ is a chain complex.*

Proof (adapted from [3]). Let $\gamma = (v_0, \ldots, v_{i+1})$ be a $(i{+}1)$-dimensional simplex of X. Let $\partial_{i+1}^{\mathcal{F}}|_\gamma$ be the restriction of $\partial_{i+1}^{\mathcal{F}}$ to $\mathcal{F}(\gamma)$.

[3] see definition 1.10 for the morphism in this category

Define $\gamma_k = (v_0, \dots, \hat{v_k}, \dots, v_{i+1})$ and $\gamma_{k,l} = (v_0, \dots, \hat{v_k}, \dots, \hat{v_l}, \dots, v_{i+1})$ so that $\gamma_{k,l} \leq \gamma_k \leq \gamma$. Then

$$\begin{aligned}\partial_i^{\mathcal{F}} \circ \partial_{i+1}^{\mathcal{F}}|_\gamma &= \sum_{k=0}^{i+1} (-1)^k \partial_i^{\mathcal{F}} \circ \mathcal{F}(\gamma \geq \gamma_k) \\ &= \sum_{k=0}^{i+1} \sum_{l=0}^{k-1} (-1)^k (-1)^l \mathcal{F}(\gamma \geq \gamma_{l,k}) \\ &+ \sum_{k=0}^{i+1} \sum_{l=k+1}^{i+1} (-1)^k (-1)^{l-1} \mathcal{F}(\gamma \geq \gamma_{k,l}) \\ &= 0\ .\end{aligned}$$

The last line comes from noting that $\gamma_{l,k} = \gamma_{k,l}$ hence the two sums cancel. $\square$

Definition 1.6. As $(\mathrm{C}_\bullet(X;\mathcal{F}), \partial_\bullet^{\mathcal{F}})$ is a chain complex we can form its homology $\mathrm{H}_\bullet(X;\mathcal{F})$ by

$$\mathrm{H}_i(X;\mathcal{F}) := \frac{\ker \partial_{i+1}^{\mathcal{F}}}{\operatorname{im} \partial_i^{\mathcal{F}}}$$

for $i \geq 0$. If $\mathrm{H}_i(X;\mathcal{F})$ is trivial for all positive i, and $\mathbb{F}$ for $i = 0$, we say $\mathcal{F}$ is acyclic[4].

Here are some examples of cosheafs over simplicial complexes:

Example 1.7 (Zero)**.** The zero cosheaf $\mathbf{0}_X$ is the cosheaf with the zero vector space as its stalks and every map between the stalks is the zero map. Note that the zero map is the identity on the zero vector space.

Example 1.8 (Constant)**.** The constant cosheaf $\mathcal{R}_X$ is the cosheaf with $\mathbb{F}$ as its stalks and every map between the stalks is the identity map. Note that the homology of X with coefficients in $\mathcal{R}_X$ is exactly the simplicial homology of X.

Example 1.9 (Leray)**.** cf. [4]. Suppose we have a simplicial map $f : X \to Y$. Then for each simplex $\sigma \in Y$ define the star of the simplex σ as the set of all cofaces of σ in Y:

$$\mathrm{St}_Y(\sigma) := \{\ \tau \in Y : \tau \geq \sigma\}.$$

Then St is contravariant: if $\sigma \leq \tau$ then $\mathrm{St}_Y(\sigma) \supseteq \mathrm{St}_Y(\tau)$ and so we have also have a simplicial inclusion $\iota : f^{-1}(\mathrm{St}_Y(\tau)) \to f^{-1}(\mathrm{St}_Y(\sigma))$. Hence for each $i \geq 0$ we have an induced map on simplicial homology

$$\iota_* : \mathrm{H}_i(f^{-1}(\mathrm{St}_Y(\tau)); \mathbb{F}) \to \mathrm{H}_i(f^{-1}(\mathrm{St}_Y(\sigma)); \mathbb{F})$$

which is functorial. Then define the ith Leray cosheaf of f as

$$\mathcal{L}_i^f(\sigma) := \mathrm{H}_i(f^{-1}(\mathrm{St}_Y(\sigma)); \mathbb{F})$$

with the maps betweens stalks ι_* as above.

[4]i.e. its reduced homology is trivial in all degrees.

1.2 Task II

Given two cosheaves $\mathcal{F}$, $\mathcal{G}$ on X we would like to have a notion of a morphism between them. A cosheaf is defined by two collections of data: the stalks, and the corestriction maps between them. As the stalks are just vector spaces we could define a morphism of cosheaves as a collection of linear maps between their stalks on the same simplex. However this does not consider any of the corestrictions. For example, let $\mathcal{F}$ be the cosheaf on X which is has $\mathbb{F}$ for each stalk, each corestriction from a simplex to itself the identity map, and the rest of the corestrictions the zero map. Then we could define a 'morphism' from $\mathcal{F}$ to $\mathcal{R}_X$ by the collection $(\mathrm{id}_{\mathbb{F}} : \mathcal{F}(\sigma) \to \mathcal{R}_X(\sigma))_{\sigma \in X}$. This 'morphism' would be invertible yet would tell us nothing about the relationship between the two cosheaves. The homology of X with coefficients in $\mathcal{R}_X$ is the ordinary simplicial homology of X. On the other hand the homology with coefficients in $\mathcal{F}$ in degree i is $\mathbb{F}^{n_i}$ where n_i is the number of simplices of X of dimension i. In fact, from the viewpoint of category theory, the actual important information contained in a cosheaf is in its corestrictions. Cosheaves are functors so luckily we already have the notion of a *natural transformation* between two functors.

Definition 1.10. (Morphism of Cosheaves) Let $\mathcal{F}$ and $\mathcal{G}$ be cosheaves over X. A collection of linear maps $\eta = \{\eta_\sigma \ : \mathcal{F}(\sigma) \to \mathcal{G}(\sigma)\}_{\sigma \in X}$ is called a *morphism of cosheaves* $\mathcal{F} \xrightarrow{\eta} \mathcal{G}$ if for any $\sigma \leq \tau \in X$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(\tau) & \xrightarrow{\mathcal{F}(\tau \geq \sigma)} & \mathcal{F}(\sigma) \\ \downarrow{\scriptstyle \eta_\tau} & & \downarrow{\scriptstyle \eta_\sigma} \\ \mathcal{G}(\tau) & \xrightarrow{\mathcal{G}(\tau \geq \sigma)} & \mathcal{G}(\sigma) \end{array} .$$

The linear maps η_σ are called the *components* of η.

To fully describe the category $\mathbf{CoShv}_X$ we need to define composition. This is done component-wise and clearly satifies the diagram in 1.10.

Definition 1.11. Given two morphisms of cosheaves $\mathcal{F} \xrightarrow{\eta} \mathcal{G} \xrightarrow{\epsilon} \mathcal{H}$ over X we define their (vertical) composition as the morphism $\epsilon \circ \eta : \mathcal{F} \to \mathcal{H}$ with components $(\epsilon \circ \eta)_\sigma = \epsilon_\sigma \circ \eta_\sigma$.

In a general, a *monomorphism* is a morphism f such that for any other morphisms a, b if $f \circ a = f \circ b$ then we have $a = b$. Certainly then in $\mathbf{CoShv}_X$, if η is a monomorphism its components are also monomorphisms, and a monomorphism in the category of vector spaces is just an injective linear map. The converse is also true. Suppose η is a morphism of cosheaves over X such that its components are all injective. Then suppose α, β are two other morpisms such that $\eta \circ \alpha = \eta \circ \beta$. Then for any $\sigma \in X$ we have that $\eta_\sigma \circ \alpha_\sigma = \eta_\sigma \circ \beta_\sigma$ hence as η_σ is injective we have that $\alpha_\sigma = \beta_\sigma$ and thus that $\alpha = \beta$. Therefore a morphism of cosheaves is a monomorphism if and only if its components are injective. Dually, we have that a morphism of cosheaves is an *epimorphism*[5] if and only if its components are surjective.

Furthermore, $\eta \circ \epsilon = \epsilon \circ \eta = \mathrm{id}$ if and only if $\eta_\sigma \circ \epsilon_\sigma = \epsilon_\sigma \circ \eta_\sigma = \mathrm{id}_\sigma$ for all $\sigma \in X$. Thus $\eta : \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if its components are invertible, and its inverse is the morphism η^{-1} with components η_σ^{-1} . This is a morphism as for $\sigma \leq \tau \in X$

$$\eta_\sigma \circ \mathcal{F}(\tau \geq \sigma) = \mathcal{G}(\tau \geq \sigma) \circ \eta_\tau$$

implies that

$$\mathcal{F}(\tau \geq \sigma) \circ \eta_\tau^{-1} = \eta_\sigma^{-1} \circ \mathcal{G}(\tau \geq \sigma) \ .$$

Definition 1.12. Suppose $\mathcal{F}, \mathcal{H}$ are cosheaves over X. We say $\mathcal{H}$ is a subcosheaf of $\mathcal{F}$ if there exists a monomorphism $\iota : \mathcal{H} \to \mathcal{F}$. Equivalently, $\mathcal{H}$ is a subcosheaf of $\mathcal{F}$ if the stalks $\mathcal{H}(\sigma)$ are vector subspaces of $\mathcal{F}(\sigma)$ and the maps $\mathcal{H}(\tau \geq \sigma)$ are the restrictions of $\mathcal{F}(\sigma \leq \tau)$ to these vector subspaces.

Thus to show $\ker \eta$ with stalks $\ker \eta_\sigma$ is a subcosheaf of $\mathcal{F}$ we need to show that $\mathcal{F}(\tau \geq \sigma)(\ker \eta_\tau) \subseteq \ker \eta_\sigma$. This follows directly from the functorality of η. Dually, we have $\mathrm{im}\, \eta$ with stalks $\mathrm{im}\, \eta_\sigma$ is a subcosheaf of $\mathcal{G}$.

1.3 Task III

Example 1.13 (1)**.** Let X be a triangulation of the unit interval:

$$X = \{\{0\}, \{1\}, \{0, 1\}\}$$

with its usual orientations. Then X is contractible hence its constant cosheaf $\mathcal{R}_X$ has homology

$$\mathrm{H}_i(X; \mathcal{R}_X) = \mathrm{H}_i(X; \mathbb{F}) = \begin{cases} \mathbb{F}, & \text{if } i = 0 \\ 0, & \text{otherwise} \end{cases}$$

so $\mathcal{R}_X$ is acyclic. Define $\mathcal{F} : X^{\mathrm{op}} \to \mathbf{FinVect}_{\mathbb{F}}$ by

$$\mathcal{F}(\sigma) = \mathbb{F},$$

for all $\sigma \in X$ and define the corestrictions for each $\sigma \leq \tau$ by

$$\mathcal{F}(\tau \geq \sigma) = \begin{cases} \mathrm{id}_{\mathcal{F}}(\tau), & \text{if } \sigma = \tau \\ 0, & \text{otherwise} \end{cases} .$$

Suppose $\sigma \leq \tau \leq \gamma \in X$ are simplices. Then the contravariancy of $\mathcal{F}$ only fails if either

- $\sigma \neq \gamma$ but $\sigma = \tau$ and $\tau = \gamma$. However, this is impossible by transitivity.

[5] η is an epimorphism iff for any $\alpha,\ \beta$ with $\alpha \circ \eta = \beta \circ \eta$ we have that $\alpha = \beta$.

- $\sigma = \gamma$ but $\sigma \neq \tau$ or $\tau \neq \gamma$. Again, this is impossible as $\sigma \leq \tau \leq \gamma$.

Thus as $\mathcal{F}$ also clearly satisfies the identity requirement, it is a cosheaf over X. Now consider its chain complex:

$$0 \longrightarrow \mathbb{F} \xrightarrow{\partial_1^{\mathcal{F}}} \mathbb{F} \oplus \mathbb{F} \longrightarrow 0 \ .$$

But $\partial_1^{\mathcal{F}}$ is 0, thus the homology of X with coefficients in $\mathcal{F}$ is

$$\mathrm{H}_i(X; \mathcal{F}) = \begin{cases} \mathbb{F} \oplus \mathbb{F}, & \text{if } i = 0 \\ \mathbb{F}, & \text{if } i = 1 \\ 0, & \text{otherwise} \end{cases}$$

Hence $\mathcal{F}$ is not acyclic even though $\mathcal{R}_X$ is.

Example 1.14 (2)**.** Let X be a triangulation of the circle:

$$X = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}\}$$

with its usual orientations. X is not acyclic, and so $\mathcal{R}_X$ is not acyclic. However the zero cosheaf $\mathbf{0}_X$ is acyclic as $\mathrm{C}_i(X; \mathbf{0}_X) = 0$ for all $i \geq 0$.

In the next two examples, a strict mono/epimorphism is a mono/epimorphism which is not an isomorphism.

Example 1.15 (3)**.** Let X be the triangulation of the unit interval as in example 1.13. Define a cosheaf $\mathcal{F}$ over X as follows: the stalks are $\mathcal{F}(\{0\}) = \mathbb{F} \oplus \mathbb{F}$, $\mathcal{F}(\{1\}) = \mathcal{F}(\{0,1\}) = \mathbb{F}$. For the corestrictions, $\mathcal{F}(\{0,1\} \geq \{0\})$ is inclusion into the first factor and the rest are all identity maps. As X has only 0 and 1 simplices, the fact that $\mathcal{F}$ satisfies the identity condition is sufficient enough for it to be a cosheaf.

Now we shall define the monomorphism from the constant cosheaf. Define $\iota : \mathcal{R}_X \to \mathcal{F}$ by setting $\iota_{\{0\}}$ to be inclusion into the first factor, and $\iota_{\{0,1\}}$, $\iota_{\{1\}}$ to be the identity maps.

$$\begin{tikzcd} \mathcal{R}_X & \mathbb{F} \arrow[d, "\begin{bmatrix}1\\0\end{bmatrix}"'] & \mathbb{F} \arrow[l, "\mathrm{id}"] \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \mathbb{F} \arrow[d, "\mathrm{id}"] \\ \mathcal{F} & \mathbb{F} \oplus \mathbb{F} & \mathbb{F} \arrow[l, "\begin{bmatrix}1 & 0\end{bmatrix}^{\mathrm{T}}"] \arrow[r, "\mathrm{id}"] & \mathbb{F} \end{tikzcd}$$

As in the diagram, we see that ι is in fact a morphism of cosheaves, and all its components are strictly injective hence it is a strict monomorphism of cosheaves. Now to calculate the homology of X with coefficients in $\mathcal{F}$. The chain complex is

$$0 \longrightarrow \mathbb{F} \xrightarrow{\begin{bmatrix}1 & 0 & 1\end{bmatrix}^{\mathrm{T}}} (\mathbb{F} \oplus \mathbb{F}) \oplus \mathbb{F} \longrightarrow 0$$

so we see that it has zero homology in degree 1 and

$$\frac{\mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}}{\mathbb{F} \oplus 0 \oplus \mathbb{F}} \cong \mathbb{F}$$

in degree 0. Thus, there exists a strict monomorphism $\mathcal{R}_X \underset{\ncong}{\overset{\iota}{\rightarrowtail}} \mathcal{F}$ and the homologies of X with coefficients in $\mathcal{R}_X$ and in $\mathcal{F}$ respectively are identical.

Example 1.16 (4)**.** What of the dual situation? Can we have a cosheaf $\mathcal{F}$ where X has the same homology with coefficients in $\mathcal{F}$ as with coefficients in $\mathcal{R}_X$ but there existing a strict epimorphism $\mathcal{R}_X \underset{\ncong}{\twoheadrightarrow} \mathcal{F}$ from $\mathcal{R}_X$ to $\mathcal{F}$?

In fact, no such situation can occur.

Proof. Suppose we do have a strict epimorphism π from $\mathcal{R}_X$ to $\mathcal{F}$. Then every component of π must be a (not necessarily strict) linear surjection from $\mathbb{F}$, hence every stalk of $\mathcal{F}$ is either $\mathbb{F}$ or $\{0\}$. As π is strict, at least one stalk of $\mathcal{F}$ must be trivial, otherwise we could define a left inverse for π. Suppose the stalk at $\tau \in X$ is trivial. If σ is a face of τ then as π is a morphism the diagram

$$\begin{tikzcd}
\mathcal{R}_X(\tau) = \mathbb{F} \arrow[r, "\mathrm{id}"] \arrow[d, "0"] & \mathcal{R}_X(\sigma) = \mathbb{F} \arrow[d, "\pi_\sigma"] \\
\mathcal{F}(\tau) = \{0\} \arrow[r, "0"] & \mathcal{F}(\sigma)
\end{tikzcd}$$

must commute, forcing $\pi_\sigma = 0$ and thus $\mathcal{F}(\sigma) = \{0\}$. Whence by induction we can assume τ is a 0-simplex.

Let $D_0, \dots, D_k$ be the connected components of X and let $\mathcal{G}$ be some arbitrary cosheaf over X. Then $\mathcal{G}$ restricts to a cosheaf $\mathcal{G}|_{D_i}$ on each connected component. Then as each 1-simplex is only a coface of 0-simplices in its connected component, the boundary operator $\partial_1^{\mathcal{G}}$ decomposes as $\bigoplus_{0 \le i \le k} \partial_1^{\mathcal{G}|D_i}$ thus we have that

$$\mathrm{H}_0(X; \mathcal{G}) = \bigoplus_{i=0}^{k} \mathrm{H}_0(D_i; \mathcal{G}|_{D_i}) \ .$$

This is also true for all degrees, but we will only need the 0 case. Now suppose that in some connected component D_i of X we have $\mathcal{F}(\sigma)$ non-zero for every 0-simplex $\sigma \in D_i$. We must also then have $\mathcal{F}(\tau)$ non-zero for every simplex $\tau \in D_i$ by the discussion above. Then as each π_τ must be non-zero, we have that $\mathcal{R}|_{D_i} \cong \mathcal{F}|_{D_i}$ hence $\mathrm{H}_0(D_i; \mathcal{R}|_{D_i}) \cong \mathrm{H}_0(D_i; \mathcal{F}|_{D_i})$.

Now we know that some connected component D_j of X must contain a 0-simplex such that the stalk of $\mathcal{F}$ at that simplex is zero. Let σ_0 be any 0-simplex in D_j. Then there exists some path $\tau_0, \ldots, \tau_n$ of 1-simplices in D_j such that the terminal 0-simplex of τ_m is the starting simplex of τ_{m+1} and that the terminal simplex σ of τ_n has $\mathcal{F}(\sigma) = 0$. Furthermore we can impose the restriction that apart from σ every other 0-simplex that is a face of a 1-simplex in this sequence has an $\mathcal{F}$-stalk that is $\mathbb{F}$, and that each $\mathcal{F}(\tau_m) = \mathbb{F}$. Finally, we can inductively define coefficients $a_0 = 1, a_1, \ldots, a_n \in \mathbb{F}$ such that: for each $\tau_m \geq \alpha \leq \tau_{m+1}$ we have that $a_m \mathcal{F}(\tau_m \geq \alpha) = a_{m+1}\mathcal{F}(\tau_{m+1} \geq \alpha)$. The purpose of this effort was that we now have an '$\mathcal{F}$-path' from σ_0 to σ. Thus the element (with the right ordering of the basis)

$$a = (1, a_1, \ldots, a_m, 0, \ldots, 0) \in \mathrm{C}_1(D_j; \mathcal{F}|_{D_j})$$

generates $\mathcal{F}(\sigma_0)$ under the boundary map. Hence we have that

$$\mathrm{H}_0(D_j; \mathcal{F}|_{D_j}) = 0 \ .$$

Therefore the dimension of the homology of X with coefficients in $\mathcal{F}$ is at least one less than with coefficients in $\mathcal{R}_X$. □

2 Persistent Cosheaf Homology

2.1 Task IV

We want to consider a sequence of cosheaves over X:

$$\mathcal{F}^0 \xrightarrow{\eta^0} \mathcal{F}^1 \xrightarrow{\eta^1} \ldots \xrightarrow{\eta^{i-1}} \mathcal{F}^i \xrightarrow{\eta^i} \ldots$$

Definition 2.1. We define a sequence of cosheaves over X as a set $\{\mathcal{F}^i\}_{i\in\mathbb{N}}$ of cosheaves over X with morphisms $\eta^j : \mathcal{F}^j \to \mathcal{F}^{j+1}$. Equivalently, we can consider such a sequence as a functor $(\mathbb{N}, \leq) \to \mathbf{CoShv}_X$.

As we have show, for any morphism $\eta : \mathcal{F} \to \mathcal{G}$, $\ker \eta$ and $\operatorname{im} \eta$ are subcosheaves of $\mathcal{F}$ and $\mathcal{G}$ respectively. If for each $i \geq 1$ we have $\operatorname{im} \eta_{i-1} = \ker \eta_i$ and $\ker \eta_0 = \mathbf{0}_X$ then we say a sequence $\{\mathcal{F}^i\}_{i\in\mathbb{N}}$ is exact. If we only have that $\operatorname{im} \eta_{i-1}$ is a subcosheaf of $\ker \eta_i$ then we say the sequence is a chain complex.

Proposition 2.2. *Suppose $\eta : \mathcal{F} \to \mathcal{G}$ is a morphism. Then define for $i \geq 0$*

$$C_i\eta : C_i(X; \mathcal{F}) \to C_i(X; \mathcal{G}) \text{ by } C_i\eta = \bigoplus_{dim\sigma = i} \eta_\sigma.$$

Then $C_\bullet\eta$ is a chain map.

Proof. This just follows from the fact that $\mathcal{G}(\sigma \leq \tau) \circ \iota_\tau = \iota_\sigma \circ \mathcal{F}(\sigma \leq \tau)$ and by linearity. □

Hence $\eta : \mathcal{F} \to \mathcal{G}$ induces a linear map $\mathrm{H}_\bullet \eta : \mathrm{H}_\bullet(X;\mathcal{F}) \to \mathrm{H}_\bullet(X;\mathcal{G})$. Thus given a sequence

$$\mathcal{F}^0 \xrightarrow{\eta^0} \mathcal{F}^1 \xrightarrow{\eta^1} \dots \xrightarrow{\eta^{j-1}} \mathcal{F}^j \xrightarrow{\eta^j} \dots$$

we have an induced sequence

$$\mathrm{H}_\bullet(X;\mathcal{F}^0) \xrightarrow{\mathrm{H}_\bullet \eta^0} \mathrm{H}_\bullet(X;\mathcal{F}^1) \xrightarrow{\mathrm{H}_\bullet \eta^1} \dots \xrightarrow{\mathrm{H}_\bullet \eta^{i-1}} \mathrm{H}_\bullet(X;\mathcal{F}^i) \xrightarrow{\mathrm{H}_\bullet \eta^i} \dots$$

Definition 2.3 (persistent homology)**.** Suppose we have a sequence $(\mathcal{F}^\bullet, \eta^\bullet)$ of cosheaves over X. We define the persistent homology groups of X with coefficients in this sequence as the vector spaces

$$\mathrm{H}_i^{l,k}(X;\mathcal{F}^\bullet) := \mathrm{im}(\mathrm{H}_i\eta^l \circ \cdots \circ \mathrm{H}_i\eta^{k+1} \circ \mathrm{H}_i\eta^k) \qquad \text{for } i \geq 0 \text{ and } l \geq k \geq 0.$$

2.2 Task V

Definition 2.4. Suppose we have a sequence $(\mathcal{F}^\bullet, \eta^\bullet)$ of cosheaves over X. Define a partial matching on X with respect to $(\mathcal{F}^\bullet, \eta^\bullet)$ as a set Σ of pairs of simplices $(\sigma < \tau)$ of X such that:

1. $\dim(\tau) - \dim(\sigma) = 1$.
2. $\mathcal{F}^j(\sigma < \tau)$ is invertible for all $j = 0, 1, \ldots$.
3. neither σ nor τ appear in any other pair of Σ.

Thus Σ is a partial matching for each individual $\mathcal{F}^j$ in the sense of [2].

Definition 2.5. Given a partial matching Σ, a Σ gradient path is a sequence $\mathfrak{p}$ of simplices $(\sigma_0, \tau_0, \sigma_1, \tau_1, \ldots, \sigma_k, \tau_k)$ of X such that

$$\underbrace{\sigma_0 < \tau_0}_{\in\Sigma} > \underbrace{\sigma_1 < \tau_1}_{\in\Sigma} > \cdots > \underbrace{\sigma_k < \tau_k}_{\in\Sigma}$$

with all σ_i having the same dimension, i.e. $\dim \tau_i - \dim \sigma_{i-1} = 1$.

Definition 2.6. We say a partial matching Σ is acyclic if there are no Σ gradient paths

$$\mathfrak{p} = (\sigma_0, \tau_0, \sigma_1, \tau_1, \ldots, \sigma_k, \tau_k)$$

with $k > 0$ and σ_0 a face of τ_k.

Definition 2.7. Given an acyclic partial matching Σ on X, we say a simplex σ is Σ-critical if it does not appear in any pair of Σ.

Example 2.8. We can retrieve an acyclic partial matching for a filtration of a simplicial complex as in the sense of [1, 4].

Suppose X is a subcomplex of Y Then we can extend the constant cosheaf on X to a cosheaf of Y by the pushforward. Define the cosheaf $\mathcal{R}_{X*}$ on Y by

$$\mathcal{R}_X(\sigma) = \begin{cases} \mathbb{F}, & \text{if } \sigma \in X \\ 0 & \text{otherwise} \end{cases}.$$

The maps $\mathcal{R}_{X*}(\tau \geq \sigma)$ are $\mathrm{id}_{\mathbb{F}}$ if both σ and τ are in X and zero otherwise. We also have a morphism $\iota : \mathcal{R}_{X*} \to \mathcal{R}_Y$ with components identity maps.

Suppose then that we have a filtration on finite simplicial complex X:

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$$

Then we have a sequence of cosheaves on X

$$\mathcal{R}^0 \xrightarrow{\iota_0} \mathcal{R}^1 \xrightarrow{\iota_1} \cdots \xrightarrow{\iota_{k-1}} \mathcal{R}^k \ .$$

Define for $\sigma \in X$

$$b(\sigma) = \min\{i \geq 0 : \mathcal{R}^i(\sigma) = \mathbb{F}\}.$$

Then as each X_i is a subcomplex, if $\sigma \leq \tau \in X_i$ then $\sigma \in X_i$ hence $b(\sigma) \leq b(\tau)$. Suppose then we have an acyclic partial matching Σ in the sense of definition 2.4. Then if $(\sigma < \tau) \in \Sigma$ by definition we have $\mathcal{R}^i(\tau > \sigma)$ invertible for all $i \geq 0$ thus we must in fact have $b(\sigma) = b(\tau)$. Hence Σ is a acyclic partial matching compatible with the filtration in the sense of [1, 4].

2.3 Task VI

Definition 2.9. Given a Σ gradient path $\mathfrak{p} = (\sigma_0, \tau_0, \sigma_1, \tau_1, \ldots, \sigma_k, \tau_k)$ we define its j-multiplicity as

$$\begin{aligned} m^j(\mathfrak{p}) = [\tau_k : \sigma_k]\mathcal{F}^j(\tau_k > \sigma_k)^{-1} \circ [\tau_{k-1} : \sigma_k]\mathcal{F}^j(\tau k - 1 > \sigma_k)^{-1} \circ \cdots \\ \cdots \circ [\tau_0 : \sigma_1]\mathcal{F}^j(\tau_0 > \sigma_1) \circ [\tau_0 : \sigma_0]\mathcal{F}^j(\tau_0 > \sigma_0)^{-1} \end{aligned}$$

which is a linear map $m^j(\mathfrak{p}) : \mathcal{F}^i(\sigma_0) \to \mathcal{F}^i(\tau_k)$.

Definition 2.10. Suppose γ and δ are two Σ-critical cells of dimensions i and $i-1$ respectively. We define the (i, j)th Morse boundary operator between them as

$$\begin{aligned} e^j_{\gamma,\delta} &= [\gamma : \delta]\mathcal{F}^j(\gamma > \delta) \\ &+ \sum_{\substack{\mathfrak{p}=(\sigma_0,\ldots,\tau_k) \\ \mathfrak{p} \text{ a } \Sigma \text{ gradient path} \\ \gamma>\sigma_0 \text{ and } \tau_k>\delta}} [\tau_k : \delta]\mathcal{F}^j(\tau_k > \delta) \circ m^j(\mathfrak{p}) \circ [\gamma : \sigma_0]\mathcal{F}^j(\gamma > \sigma_0) \end{aligned}$$

which is a linear map $e^j_{\gamma,\delta} : \mathcal{F}^j(\gamma) \to \mathcal{F}^j(\delta)$.

Definition 2.11. The jth Morse chain complex associated to Σ is the sequence M^j of vector spaces

$$\cdots \xrightarrow{e^j_{i+1}} \mathrm{M}^j_i \xrightarrow{e^j_i} \cdots \xrightarrow{e^j_2} \mathrm{M}^j_1 \xrightarrow{e^j_1} \mathrm{M}^j_0 \longrightarrow 0$$

where

$$\mathrm{M}^j_i = \bigoplus_{\substack{\dim\sigma=i \\ \sigma \text{ critical}}} \mathcal{F}^j(\sigma)$$

and the block of e^j_i from $\mathcal{F}^j(\gamma)$ to $\mathcal{F}^j(\delta)$ is $e^j_{\gamma,\delta}$.

So far, we know that this Morse chain complex simplifies our computations for the homology of each cosheaf:

Theorem 2.12 (Sköldberg)**.** *For each i, the ith Morse chain complex associated to Σ is a bone fide chain complex with homology groups isomorphic to the homology of X with coefficients in $\mathcal{F}^i$.*

As the next theorem shows however, the acyclic partial matching we have defined is compatible with the sequence and computes its persistence.

Definition 2.13. For each $i, j \geq 0$ define $\hat{\eta}^j_i : \mathrm{M}^j_i \to \mathrm{M}^{j+1}_i$ by

$$\hat{\eta}^j_i = \bigoplus_{\substack{\dim\sigma=j \\ \sigma \text{ critical}}} \eta^j_\sigma \quad .$$

Theorem 2.14. *(a) $\hat{\eta}^j = (\hat{\eta}^j_i : \mathrm{M}^j_i \to \mathrm{M}^{j+1}_i)_{i\geq 0}$ is a chain map from M^j to M^{j+1} for each $j \geq 0$, hence we have an induced sequence*

$$\mathrm{H}_\bullet(\mathrm{M}^0) \xrightarrow{\mathrm{H}_\bullet\hat{\eta}^0} \mathrm{H}_\bullet(\mathrm{M}^1) \xrightarrow{\mathrm{H}_\bullet\hat{\eta}^1} \cdots \xrightarrow{\mathrm{H}_\bullet\hat{\eta}^{j-1}} \mathrm{H}_\bullet(\mathrm{M}^j) \xrightarrow{\mathrm{H}_\bullet\hat{\eta}^j} \cdots .$$

(b) The persistent homology groups of this sequence of Morse complexes are isomorphic to the persistent homology groups of X with coefficients in the sequence $(\mathcal{F}^\bullet, \eta^\bullet)$:

$$\mathrm{H}^{l,k}_i(\mathrm{M}^\bullet) := \mathrm{im}(\mathrm{H}_i\hat{\eta}^l \circ \cdots \circ \mathrm{H}_i\hat{\eta}^{k+1} \circ \mathrm{H}_i\hat{\eta}^k) \cong \mathrm{H}^{l,k}_i(X; \mathcal{F}^\bullet)$$

for $i \geq 0$ and $l \geq k \geq 0$.

Sketch Proof. The following proof is essentially an extension of the proof of 2.12 given in [2] to our persistent homology of cosheaves. To fully complete the proof one would have to slightly generalise our definitions of cosheaves on simplicial complexes to 'coparametrizations' of chain complexes on a poset as in [2]. Here we give a proof of the inductive step by considering a partial matching containing only one pair. For ease of notation, define $\mathrm{C}^j_i := \mathrm{C}_i(X; \mathcal{F}^j)$.

Figure 1:

$$\begin{tikzcd}
 & \mathrm{C}_i^j \arrow[rr, "\mathrm{C}_i\eta^j"] \arrow[dl, "\partial_i^j"] \arrow[dd, "\psi_i^j"] & & \mathrm{C}_i^{j+1} \arrow[dl] \arrow[dd] \\
\mathrm{C}_{i-1}^j \arrow[rr] \arrow[dd] & & \mathrm{C}_{i-1}^{j+1} \arrow[dd] & \\
 & \mathrm{M}_i^j \arrow[rr, "\hat{\eta}_i^j"] \arrow[dl, "e_i^j"] & & \mathrm{M}_i^{j+1} \arrow[dl] \\
\mathrm{M}_{i-1}^j \arrow[rr] & & \mathrm{M}_{i-1}^{j+1} &
\end{tikzcd}$$

Figure 2:

(1)

$$\begin{tikzcd}
 & \mathrm{C}_i^j \arrow[d] & \\
 & \mathrm{M}_i^j \arrow[dl] & \\
\mathrm{M}_{i-1}^j \arrow[rr] & & \mathrm{M}_{i-1}^{j+1}
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & \mathrm{C}_i^j \arrow[dl] \\
\mathrm{C}_{i-1}^j \arrow[d] & \\
\mathrm{M}_{i-1}^j \arrow[r] & \mathrm{M}_{i-1}^{j+1}
\end{tikzcd}$$

(3)

$$\begin{tikzcd}
 & \mathrm{C}_i^j \arrow[dl] & \\
\mathrm{C}_{i-1}^j \arrow[rr] & & C_{i-1}^{j+1} \arrow[d] \\
 & & \mathrm{M}_{i-1}^{j+1}
\end{tikzcd}$$

(4)

$$\begin{tikzcd}
\mathrm{C}_i^j \arrow[rr] & & \mathrm{C}_i^{j+1} \arrow[dl] \\
 & C_{i-1}^{j+1} \arrow[d] & \\
 & \mathrm{M}_{i-1}^{j+1} &
\end{tikzcd}$$

(5)

$$\begin{tikzcd}
\mathrm{C}_i^j \arrow[r] & \mathrm{C}_i^{j+1} \arrow[d] \\
 & \mathrm{M}_{i-1}^j \arrow[dl] \\
\mathrm{M}_{i-1}^{j+1} &
\end{tikzcd}$$

(6)

$$\begin{tikzcd}
\mathrm{C}_i^j \arrow[d] & & \\
\mathrm{M}_i^j \arrow[rr] & & \mathrm{M}_i^{j+1} \arrow[dl] \\
 & \mathrm{M}_{i-1}^{j+1} &
\end{tikzcd}$$

Let our partial matching Σ be just one pair $x^* < y^*$ where $\dim x = i$. Then by definition, Σ is a acyclic partial matching for each $\mathcal{F}^j$ so we can use the results of [2]. Thus define a map $\psi_i^j : \mathrm{C}_i^j \to \mathrm{M}_i^j$ by the block form

$$\psi_{z,\tilde{z}}^j = \begin{cases} [y^* : \tilde{z}][y^* : x^*]\mathcal{F}^j(y^* > \tilde{z}) \circ \mathcal{F}^j(y^* > x^*)^{-1} & \text{if } z = x^*, \\ \mathrm{id}_{\mathcal{F}^i(z)} & \text{if } z = \tilde{z}, \\ 0 & \text{otherwise} \end{cases}$$

and define $\psi_k^j : \mathrm{C}_k^j \to \mathrm{M}_k^j$ as the identity map for $k \neq i$. Then as in the dual proofs in section 3 of [2], ψ^j is in fact a chain map and there is another chain map $\phi^j : \mathrm{M}^j \to \mathrm{C}^j$ such that $\psi^j \circ \phi j = \mathrm{id}_{\mathrm{M}^j}$ and $\phi^j \circ \psi_j$ is chain homotopic to $\mathrm{id}_{\mathrm{C}^j}$. Hence, as in theorem 2.12, we have for each $j \geq 0$

$$\mathrm{H}_i(\mathrm{M}^j) \cong \mathrm{H}_i(X; \mathcal{F}^j)$$

Now we wish to extend this to show that:

(a) The Morse complexes M^j are compatible, i.e. for each j the collection of maps in definition 2.13 is actually a chain map.

(b) The induced sequence of this sequence of Morse complexes give us the persistent homology groups of X with coefficients in our sequence of cosheaves.

Consider figure 1. To show $\hat{\eta}^j$ is a chain map we need to show the 'pink' square commutes. Firstly however, we need to prove that the 'green' square commutes. We only need to prove the case $z = x^*$ as the rest are all identity maps. Let $\tilde{z}$ be a dimension i critical simplex. Consider the block $\psi_{x^*,\tilde{z}}^j$. We need to show that $\eta_{\tilde{z}}^j \circ \psi_{x^*,\tilde{z}}^j = \psi_{x^*,\tilde{z}}^{j+1} \circ \eta_{x^*}^j$. We have

$$\begin{aligned} &\eta_{\tilde{z}}^j \circ [y^* : \tilde{z}][y^* : x^*]\mathcal{F}^j(y^* > \tilde{z}) \circ \mathcal{F}^j(y^* > x^*)^{-1} \\ &= [y^* : \tilde{z}][y^* : x^*]\mathcal{F}^{j+1}(y* > \tilde{z}) \circ \eta_{y^*}^j \circ \mathcal{F}^j(y^* > x^*)^{-1} \\ &= [y^* : \tilde{z}][y^* : x^*]\mathcal{F}^{j+1}(y* > \tilde{z}) \circ \mathcal{F}^{j+1}(y^* > x^*)^{-1} \circ \eta_{x^*}^j \quad . \end{aligned}$$

Hence the 'green' square commutes. Now to prove that the 'pink' square commutes, consider the composition

$$\mathrm{C}_i^j \xrightarrow{\psi_i^j} \mathrm{M}_i^j \xrightarrow{e_i^j} \mathrm{M}_{i-1}^j \xrightarrow{\hat{\eta}_{i-1}^j} \mathrm{M}_{j+1}^{i-1}$$

and the sequence of 'moves' in figure 2.

- Composition (2) is equal to (1) by commutativity of the 'blue' square.
- Composition (3) is equal to (2) by commutativity of the square opposite the 'green' square.
- Composition (4) is equal to (3) by commutativity of the square opposite the 'pink' square.

- Composition (5) is equal to (4) by commutativity of the square opposite the 'blue' square.
- Finally, Compositon (6) is equal to (5) by commutativity of the 'green' square.

Thus $\hat{\eta}^j_{i-1} \circ e^j_i \circ \psi^j_i = e^{j+1}_i \circ \hat{\eta}^j_i \circ \psi^j_i$. Importantly however, ψ^j_i is a surjection thus we can cancel it from the right (i.e. it has a right inverse namely ϕ^j_i) hence $\hat{\eta}^j_{i-1} \circ e^j_i = e^{j+1}_i \circ \hat{\eta}^j_i$. Thus the 'pink' square commutes and we have proven (a).

We have $\mathrm{H}_i\phi^l = (\mathrm{H}_i\psi^l)^{-1}$ and $\mathrm{H}_i\hat{\eta}^l \circ \mathrm{H}_i\psi^l = \mathrm{H}\psi^{l+1} \circ \mathrm{H}_i\eta^l$ hence $\mathrm{H}_i\hat{\eta}^l = \mathrm{H}\psi^{l+1} \circ \mathrm{H}_i\eta^l \circ \mathrm{H}_i\phi^l$. Thus by induction

$$\mathrm{H}_i\hat{\eta}^l \circ \cdots \circ \mathrm{H}_i\hat{\eta}^k = \mathrm{H}\psi^{l+1} \circ \mathrm{H}_i\eta^l \circ \cdots \circ \mathrm{H}_i\eta^k \circ \mathrm{H}_i\phi^k.$$

Therefore, $\mathrm{H}\psi^{l+1}$ gives an isomorphism between $\mathrm{H}^{l,k}_i(X;\mathcal{F}^\bullet)$ and $\mathrm{H}^{l,k}_i(\mathrm{M}^\bullet)$, and we have proved (b). $\square$

In conclusion, we have described what the obscure sounding 'persistent homology of sequences of cosheaves over a finite simplicial complex' is and given a (sketch) proof of how to extend discrete Morse theory to simplify its computation. The question that remains however is what is an efficient algorithm for constructing a good acyclic partial matching? Indeed, [2] gives an efficient algorithm for finding a matching for each cosheaf in the sequence but it might not be the case that this matching is the 'most' efficient over the whole sequence.

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